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GEOMETRICAL IMPLICATIONS OF CERTAIN INFINITE DIMENSIONAL DECOMPOSITIONS

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ABSTRACT. We investigate the connections between the "global" structure of a Banach space (i.e. the existence of certain finite and infinite dimensional decompositions) and the geometrical properties of the closed convex bounded subsets of such a space (i.e. the existence of extremal and other topologically distinguished points). The global structures of various—supposedly pathological—examples of Banach spaces constructed by R. C. James turn out to be more "universal" than expected. For instance *James-tree-type* (resp. *James-matrix-type*) *decompositions* characterize Banach spaces with the *Point of Continuity Property* (resp. the *Radon-Nikodým Property*). Moreover, the *Convex Point of Continuity Property* is stable under the formation of *James-infinitely branching tree-type "sums"* of infinite dimensional factors. We also give several counterexamples to various questions relating some topological and geometrical concepts in Banach spaces.

INTRODUCTION

One of the basic problems in infinite dimensional Banach space theory is the study of the interrelations between the "global" and the geometrical structures of a Banach space. Here is an example of such a—still unresolved—problem:

Given a Banach space X with a Schauder basis, what global conditions on such a basis insure that every closed bounded subset of X has an extreme point?

A well-known sufficient condition is that the basis be *boundedly complete* since in such a case, X is isomorphic to a separable dual by a classical result of James [LT, p. 9] and these spaces have a rich extremal structure by a result of Bessaga-Pelczynski [DU, p. 209]. However the pathological $\mathscr{L}_\infty$-spaces constructed by Bourgain-Delbaen [BD] show that the situation is much more complicated and that the above-mentioned condition on the basis is far from being necessary. More recently, Bourgain and Rosenthal [BR1] initiated the study of some geometrical structures that are implied by much weaker types of finite dimensional decompositions. They introduce the notions of "l^1-skipped" and "boundedly complete skipped" blocking finite dimensional decompositions

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(see definitions below). They show that the first type of decompositions implies the *Radon-Nikodým Property* (R.N.P) (equivalently, every bounded closed set has an extreme point) while the second type implies the *Point of Continuity Property* (P.C.P) (equivalently, every bounded closed set has a point of weak to norm continuity). In [GM1] the two first-named authors showed that in the latter case the converse also holds true. Actually, the following theorem was established. We refer to the end of the introduction for the relevant definitions.

Theorem (A) [GM1]. *For a separable Banach space X, the following assertions are equivalent:*

(1) *X has the Point of Continuity Property (P.C.P).*
(2) *X has a boundedly complete skipped blocking finite dimensional decomposition.*
(3) *There exists a compact G_δ-embedding of X into Hilbert space.*

Our aim in §I of this paper is to make this result more precise and to give its analogue in the case of Banach spaces with the *Radon-Nikodým Property.* For that, we introduce the *James-tree type* (JT-type) and the *James-matrix type* (JM-type) decompositions of a Banach space into finite dimensional components. The terminology is suggested by the facts (proved in §I) that spaces with the P.C.P have a structure very much like the predual of James-tree space [LS] while spaces with the R.N.P have a structure similar to a space of infinite matrices on which a James space-like norm is defined.

In §II we start the study of various types of decompositions of Banach spaces into *infinite dimensional factors.* Our interest in this setting was triggered by the space J_*T_∞ (or B_∞) constructed in [GM1]. It is shown there that this space fails to have the P.C.P even though it has:

(α) A *boundedly complete skipped blocking decomposition into (infinite dimensional) complemented Hilbert spaces.*
(β) A *(noncompact) G_δ-embedding into Hilbert space.*

Moreover, it was shown in [GMS] that this space has the so-called:

(γ) *Convex Point of Continuity Property* (C.P.C.P): that is every closed bounded and *convex* set has a point of weak to norm continuity .

So the question arose whether a result analogous to Theorem (A) and relating (α), (β) and (γ) might hold in full generality. In other words, what can be said when the factor spaces are well behaved but infinite dimensional or when the G_δ-embedding is not supposed to be compact. Do they correspond to the case where only *convex* (bounded closed) sets have points of continuity?

Unfortunately, the situation turned out to be much more complicated in this "infinite dimensional" setting and a large part of this paper consists of various counterexamples showing that these assertions ((α), (β) and (γ)) are essentially incomparable. For instance, in §VI we construct a Banach space denoted S_*T_∞ that verifies (α) and (β) but not (γ) which means that even the C.P.C.P is not stable under "boundedly complete skipped blocking sums".

On the other hand, in §II we give two positive results: First, the stability of the Radon-Nikodým property under "l^1-skipped blocking sums" even when the factors are infinite dimensional. This is an easy extension of the Bourgain-Rosenthal result mentioned above [BR1]. Secondly, we consider the geometrical concept of *strong regularity* studied in [GGMS], i.e. every closed convex bounded set has convex combinations of slices of arbitrarily small diameter. This notion is strictly weaker than the C.P.C.P (see §VI) but unlike the other related notions, it has the advantage of being stable under "boundedly complete skipped blocking sums".

While searching for the type of decomposition that is compatible with the C.P.C.P, we were led to the apparently stronger concept of *Uniform Convex Point of Continuity Property* (Uniform C.P.C.P). This notion is introduced and studied in §III where we show, for instance, that it is implied by the P.C.P. We then prove, in §IV, that it is actually stable under James-type (JT_∞-type) decompositions which again is a strengthening of the notion of boundedly complete skipped blocking decomposition. As a consequence we obtain that all preduals $J_*T_{\infty,n}$ of the iterated James-tree spaces constructed in [GM2] have the uniform C.P.C.P, even though $J_*T_{\infty,2}$ does not G_δ-embed in l_2 and cannot have a boundedly complete skipped blocking decomposition into complemented reflexive spaces. Hence (γ) implies neither (α) nor (β).

In §V, we answer negatively a question of G. Godefroy by showing that the dual JT^* of (the original) James-tree space which does not have the R.N.P actually possesses the so-called w^*-*Convex Point of Continuity Property* (C_* P.C.P): every w^* compact convex subset has a point of weak* to norm continuity. In the terminology of [DGHZ], this means that JT is a *Phelps space* without being an *Asplund space.*

In §VI, we construct the space S_*T_∞ already mentioned above. It is shown, among other things, that its double dual ST_∞^* is strongly regular without having the C_* P.C.P. This means that the class of *Phelps spaces* lies strictly between the class of Asplund spaces and the class of Banach spaces not containing an isomorphic copy of l_1.

We shall now recall the definitions of the needed concepts. Most of the geometrical ones were already defined above. However, other characterizations and reformulations of these notions will be needed throughout this paper. for more details about the *Radon-Nikodým Property* (R.N.P) we refer to [DU, GM1 and GM3]. For the *Point of Continuity Property* (P.C.P) we refer to [BR1, EW and GM1]. The *Convex Point of Continuity Property* (C.P.C.P) was introduced in [B1]. The concept of *strong regularity* is developed in the memoir of [GGMS].

Here are some of the needed finite or infinite dimensional decompositions. The concepts of J-*type*, JT-*type* and JT_∞-*type decompositions* will be introduced later on and after some motivational examples.

Let $(X_n)_{n=1}^\infty$ be a sequence of closed subspaces of a Banach space X. Recall that the sequence $(X_n)_{n=1}^\infty$ is called a *Schauder decomposition* of X if every

$x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for all n. The decomposition is *boundedly complete* if, for every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$ the series $\sum_{n=1}^{\infty} x_n$ converges whenever $(\|\sum_{i=1}^{n} x_i\|)_{n=1}^{\infty}$ is bounded. The sequence $(X_n)_{n=1}^{\infty}$ is an *l^1-decomposition* if there is some $\delta > 0$ with $\|\sum_{i=1}^{n} x_i\| \geq \delta \sum_{i=1}^{n} \|x_i\|$ for every finite sequence $(x_i)_{i=1}^{n}$ with $x_i \in X_i$.

For a sequence $(X_n)_{n=1}^{\infty}$ of closed subspaces of X and $m \leq n$ denote by $X[m, n]$ the closure of the span of $(X_i)_{i=m}^{n}$, and $[X_n]_{n \in A}$ the closure of the span of $(X_n)_{n \in A}$ for $A \subset N$. With this notation we recall that the a sequence $(X_n)_{n=1}^{\infty}$ of closed (not necessarily finite-dimensional) subspaces of a Banach space X is a *boundedly complete* (resp. l^1) *skipped blocking decomposition of* X provided:

(i) $X = [X_n]_{n=1}^{\infty}$,

(ii) $X_n \cap [X_m]_{m \neq n} = \{0\}$ for $n \in \mathbf{N}$, and

(iii) if $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ are sequences in $\mathbf{N}$ such that $m_k < n_k + 1 < m_{k+1}$ for all k, then the sequence $Y_k = X[m_k, n_k]$ is a boundedly complete (resp. l^1) Schauder decomposition of its closed linear span.

If the subspaces X_n are also complemented in X we shall say that $(X_n)_{n=1}^{\infty}$ is a *complemented boundedly complete* (resp. l^1) *skipped blocking decomposition of* X.

Concerning the topological properties we recall that a one-to-one bounded linear operator $T: X \to Y$ is said to be a *G_δ-embedding* (resp. an *H_δ-embedding*) if for any closed bounded subset F of X we have that $\overline{T(F)} \setminus T(F) = \bigcup_n K_n$, where each K_n is norm closed (resp. closed and convex). T is said to be a *semiembedding* (*resp. a Tauberian embedding*) if the image of the unit ball (resp. every closed convex bounded subset) of X is closed in Y. For more details about these notions we refer to [BR2 and GM1].

A *slice* of a bounded subset C of a Banach space X is a set of the form $S = S(x^*, C, \alpha) = \{x \in C : x^*(x) > \sup_C x^* - \alpha\}$, where $x^* \in X^*$ and $\alpha > 0$.

By an X-valued *martingale* $(M_n)_{n=1}^{\infty}$ we mean a sequence of finitely valued random variables adapted to an increasing sequence of σ-fields $(\mathscr{F}_n)_{n=1}^{\infty}$ on $[0, 1]$ equipped with Lebesgue measure λ, such that that for each n, $\int_A M_{n+1}\, d\lambda = \int_A M_n\, d\lambda$ for all A in $\mathscr{F}_n$.

Most of our examples will be spaces of functions defined on trees. We shall distinguish the tree $T = \bigcup_{k=1}^{\infty} \{0, 1\}^k$ with finitely many branching points and the tree $T_\infty = \bigcup_{k=0}^{\infty} \mathbf{N}^k$ with infinitely many branching points. If $t = (n_1, n_2, \dots, n_k) \in T$ (or T_∞) we denote $|t| = k$ and for $j \leq k$ we set $t|_j = (n_1, n_2, \dots, n_j)$. The partial order on T (or T_∞) is defined by $s \leq t$ if $|s| \leq |t|$ and $s = t_{|s|}$. For each element $(n_k)_k$ in $\{0, 1\}^{\mathbf{N}}$ (or $\mathbf{N}^{\mathbf{N}}$) we associate the *branch* $\gamma = \{\phi, (n_1), (n_1, n_2), \dots, (n_1, n_2, \dots, n_k), \dots\} \subset T$ (or T_∞). Set $\gamma|_k = (n_1, n_2, \dots, n_k) \in T$ (resp. T_∞). A *segment* in T (or T_∞) is a subset $\{t_0, t_2, \dots, t_n\}$ of T (resp. T_∞) with $t_0 < t_1 < \cdots < t_n$ and $|t_k| = |t_0| + k$ for $k = 0, 1, \dots, n$.

For unexplained notations or terminology we refer the reader to the books [DU and LT].

I. James-matrix and James-tree type finite dimensional decompositions

Recall the definition of James-tree space JT (cf. [LS]): It is the completion of the space of simple functions on the binary tree $T = \bigcup_{n=0}^{\infty}\{0,1\}^n$ equipped with the norm

$$\|x\|_{JT} = \sup\left\{ \left(\sum_{i=1}^{m} \left| \sum_{t\in S_i} x(t) \right|^2 \right)^{1/2} : S_1, \dots, S_m \text{ disjoint segments of } T \right\}.$$

Note that JT has a (unique) predual $B = J_*T$ which equals the closed span of the coefficient functionals $(e_t^*)_{t\in T}$ in JT^*. The space B is—as we shall see in a sense made precise below—the typical example of a separable Banach space having P.C.P but failing R.N.P.

On the other hand, in order to get a typical example of a separable Banach space having R.N.P, we need to consider the space $JM = l^2(J)$, where J is the usual James space [LT, p. 25] which is obtained by completing the space of simple functions on $\mathbf{N}$ with respect to the norm

$$\|(x_n)_{n=1}^{\infty}\|_J = \sup\left\{ \left(\sum_{i=1}^{m} \left| \sum_{n=n_i}^{n_{i+1}-1} x_n \right|^2 \right)^{1/2} : 1 \le n_1 < n_2 < \cdots < n_{m+1}, k \in \mathbf{N} \right\},$$

The letters JM stand for "James-matrix" space as one easily checks that we may represent JM as the completion of the space of finitely supported lower triangular matrices $((x_{j,n})_{j=n}^{\infty})_{n=1}^{\infty}$ equipped with the norm

$$\|((x_{j,n})_{j=n}^{\infty})_{n=1}^{\infty}\|_{JM} = \sup\left\{ \left(\sum_{i=1}^{m} \left| \sum_{(j,n)\in S_i} x_{j,n} \right|^2 \right)^{1/2} : S_1, \dots, S_m \text{ disjoint segments of } M \right\},$$

where $M = \{(j,n) : n \le j;\ j, n \in \mathbf{N}\} \subseteq \mathbf{N}\times\mathbf{N}$ and where a segment S of M is a vertical strip of the form $S = \{(j,n), (j+1,n), \dots, (j+k,n)\}$ for some integers $n \ge 1$, $j \ge n$ and $k \ge 0$.

Clearly JM has a (unique) predual $J_*M = l^2(J_*)$ formed by the closed span of the coefficient functionals $((e_{j,n})_{j=n}^{\infty})_{n=1}^{\infty}$ in JM^*. Note that JM^*/J_*M is isometric to the separable, infinite-dimensional Hilbert space l^2. In particular

J_*M has the Radon-Nikodým property as it is a subspace of the separable dual JM^*.

The spaces J_*T and J_*M have obvious finite dimensional decompositions: In the case of J_*T the 2^n-dimensional spaces $X_n = \operatorname{span}\{e_t : |t| = n\}$, $n = 0, 1, 2, \ldots$, and in the case of J_*M, the n-dimensional spaces $X_n = \operatorname{span}\{e_{n,k} : 1 \le k \le n\}$, $n = 1, 2, \ldots$.

These finite dimensional decompositions have special properties that we shall isolate in the subsequent definition.

The main purpose of this section is to show that the existence of decompositions of the above types actually characterizes separable Banach spaces with P.C.P and R.N.P respectively.

Definition I.1. A sequence $(X_n)_{n=1}^{\infty}$ of finite dimensional subspaces of a Banach space X is called

(a) a *JT-type decomposition of* X if

(i) $(X_n)_{n=1}^{\infty}$ is a boundedly complete skipped blocking decomposition of X with associated projections $p_n : X \mapsto X_n$, and

(ii) there are elements $((y_n^k)_{k=1}^{m_n})_{n=1}^{\infty}$ in X^*, $\|y_n^k\| = 1$, $y_n^k \in [X_m]_{m \ne n}^{\perp}$ such that for every $x^{**} \in X^{**}$ with coordinates $x_n = p_n^{**}(x^{**})$ the condition

$$\liminf_{n \mapsto \infty} \max_{1 \le k \le m_n} \langle x_n, y_n^k \rangle \le 0 \tag{1}$$

implies that there is an $x \in X$, $\|x\| \le \|x^{**}\|$, such that $p_n(x) = x_n$ for every $n \in \mathbf{N}$.

(b) a *JM-type decomposition of* X if it is a JT-type decomposition and in addition condition (1) may be replaced by

$$\liminf_{n \mapsto \infty} \langle x_n, y_n^k \rangle \le 0 \quad \text{for each } k \in \mathbf{N}. \tag{2}$$

Note that in both cases we then get that $\liminf_{n\to\infty} \max_{1\le k\le m_n} |\langle x_n, y_n^k\rangle| = 0$.

Remark I.2. In the case of J_*T we may choose $m_n = 2^{n+1}$ and $(y_n^k)_{k=1}^{2^{n+1}}$ to consist of the coefficient functionals $\{e_t^*\}_{|t|=n}$ and $\{-e_t^*\}_{|t|=n}$ in JT. It follows from the analysis of Lindenstrauss and Stegall [LS] that this choice fulfills the requirements of JT-type decomposition for the above defined subspaces X_n of J_*T.

In the case of J_*M we may choose $m_n = 2n$, $y_n^{2k-1} = e_{n,k}^*$ and $y_n^{2k} = -e_{n,k}^*$ for $1 \le k \le n$. In this case one easily verifies that the requirements are satisfied for the above defined subspaces X_n of J_*M.

Condition (a)(ii) in the above definition translates in the case of J_*T into the following: An element x^{**} of JT^*, identified with a function on T, does not belong to J_*T if and only if there is one of the uncountably many branches γ of T such that $\liminf_{t\in\gamma} x^*(t) > 0$ or $\liminf_{t\in\gamma} -x^*(t) > 0$. In the case of the James-matrix space, an element x^{**} of JM^*, identified with a function on $\mathbf{N}\times\mathbf{N}$, does not belong to J_*M if and only if there is one of the countably many columns $((n,k))_{n=k}^{\infty}$ of $\mathbf{N}\times\mathbf{N}$ such that $\liminf_{n\to\infty} x^{**}(n,k) > 0$ or $\liminf_{n\to\infty} -x^{**}(n,k) > 0$.

Note that we do not assume in the above definition that $(X_n)_n$ forms a Schauder decomposition for the space X. However, if they do (as in the case of the above defined decompositions for J_*T and J_*M), the situation becomes easier as we shall summarize in the following:

Proposition I.3. *Let $(X_n)_n$ be a finite dimensional monotone Schauder decomposition for a Banach space X. The following conditions are then equivalent:*

(a) *$(X_n)_n$ is a boundedly complete skipped blocking decomposition.*

(b) *$(X_n)_n$ is a JT-type decomposition.*

(c) *There exist norm-one elements $((y_n^k)_{k=1}^{m_n})_n$ in X^* with $y_n^k \in [X_m]_{m\neq n}^{\perp}$ such that whenever $(x_n)_n$ is a sequence with $x_n \in X_n$, $\sup_n \|\sum_{i=1}^n x_i\| < \infty$ and $\liminf_{n\to\infty} \max_{1\le k\le m_n}\langle x_n, y_n^k\rangle \le 0$, then the series $\sum_{i=1}^{\infty} x_i$ norm converges in X.*

Proof. Clearly (c) $\Rightarrow$ (b) $\Rightarrow$ (a). For the converse note first that if $(X_n)_n$ is a boundedly complete skipped blocking decomposition which is also a Schauder decomposition and if $x_n \in X_n$ are such that

$$\limsup_{n\mapsto\infty}\left\|\sum_{i=1}^{n} x_i\right\| < \infty \quad\text{and}\quad \liminf_{n\mapsto\infty}\|x_n\| = 0,$$

then $\sum_{i=1}^{\infty} x_i$ converges in norm. Indeed, let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence such that $\|x_{n_k}\| < 2^{-k}$ and let $y_k = \sum_{i=n_k+1}^{n_{k+1}-1} x_k$.

By definition, $\sum_{k=1}^{\infty} y_k$ converges and therefore $\sum_{k=1}^{\infty}(y_k + x_{n_k})$ converges too. By the assumption that $(X_n)_n$ is a Schauder decomposition, we infer that $\sum_{i=1}^{\infty} x_i$ also converges.

To prove that (a) implies (b) or (c) note that $[X_m]_{m\neq n}^{\perp}$ $\frac{1}{2}$-norms X_n hence it contains vectors $(y_n^k)_{k=1}^{m_n}$ which $\frac{1}{3}$-norm X_n. Given x^{**} in X^{**} with coordinates $x_n = p_n^{**}(x^{**})$ we have

$$\liminf_{n\mapsto\infty}\|x_n\| = 0 \quad\text{if and only if}\quad \liminf_{n\to\infty}\max_{1\le k\le m_n}\langle x_n y_n^k\rangle \le 0.$$

By the above argument we conclude that under this assumption $\sum_{n=1}^{\infty} x_n$ norm-converges to some $x\in X$. It is then easy to verify that $\|x\| \le \|x^{**}\|$.

Remark I.4. In the case when $(X_n)_n$ fails to be a Schauder decomposition (i.e., $(\|p_n\|)_n$ is not bounded) the situation is more complicated: If $(X_n)_n$ is a JT-type decomposition and $x^{**}\in X^{**}$ with coordinates $x_n = p_n^{**}(x^{**})$ satisfying condition I.1(1) then the norm convergence of $\sum_{n=1}^{\infty} x_n$ does not hold in general.

An easy example goes as follows: Write $X = l^2$ as $X = \bigoplus_{n=1}^{\infty} Z_n$, where the Z_n's are 2-dimensional mutually orthogonal subspaces spanning l^2. Now split Z_n into $Z_n = X_{2n-1} \oplus X_{2n}$ where X_{2n-1} and X_{2n} are one-dimensional subspaces of Z_n and such that the projections $\pi_n : Z_n \mapsto X_{2n}$ with $\ker(\pi_n) = X_{2n-1}$ satisfies $\|\pi_n\| \geq n$.

For example, one may take $Z_n = \text{span}(e_{2n-1}, e_{2n})$, $X_{2n-1} = \text{span}(e_{2n-1})$ and $X_{2n} = \text{span}(e_{2n-1} + 2^{-n} e_{2n})$, where $(e_n)_n$ is the unit vector basis of l_2. Clearly $(X_n)_{n=1}^\infty$ fails to be a Schauder decomposition of l^2 but one quickly verifies that $(X_n)_{n=1}^\infty$ is a boundedly complete skipped blocking decomposition: Indeed if $(n_k)_{k=1}^\infty$ is strictly increasing and $y_k \in \text{span}\{X_i : n_k < i < n_{k+1}\}$ then $(y_k)_{k=1}^\infty$ is an orthogonal sequence in l^2.

Clearly $(X_n)_{n=1}^\infty$ is a JT-type (even a JM-type) decomposition of l^2 since the latter is reflexive. (There is even no need for choosing the y_n^k's.) Also note that the sequence defined by $x_{2n-1} = -e_{2n-1}$ and $x_{2n} = e_{2n-1} + 2^{-n} e_{2n}$ for $n \in \mathbf{N}$ is such that the series $\sum_{n=1}^\infty x_n$ does not converge in norm.

However, we can make the following remark about the convergence of $\sum x_n$: If $(X_n)_{n=1}^\infty$ is a JT-type decomposition and $x_n = p_n^{**}(x^{**})$ satisfies I.1(a), we let Y be the subspace of X^* spanned by the annihilators $([X_m]_{m \neq n}^\perp)_{n=1}^\infty$. It is clear that $\sum_{n=1}^\infty x_n$ converges to x in the $\sigma(X, Y)$ topology. We shall see in I.14 that Y is a norming subspace of X^*.

The main result of this section is Theorem I.7. A proof can be obtained by combining the arguments in Theorem II.1 of [GM1] with those of Theorem IV.7 of [GM3]. However, we shall give here a proof based on an appropriate embedding of the Banach space X into a space with a basis (Proposition I.5) that is a refinement of a result in [JRZ] and which might be of independent interest.

Denote $\Delta = \{-1, +1\}^{\mathbf{N}}$ and, for $i \in \mathbf{N}$, $\Delta_i = \{-1, +1\}^i$. Let μ (resp. μ_i) be the normalized Haar-measure on Δ (resp. Δ_i). For $i \in \mathbf{N}$ (resp. for $i_1 \geq i_2$ in $\mathbf{N}$) denote $\pi_i : \Delta \mapsto \Delta_i$ (resp. $\pi_{i_1, i_2} : \Delta_{i_1} \mapsto \Delta_{i_2}$) as the canonical restriction map.

Proposition I.5. *Let X be a separable Banach space and let Y be a separable norming subspace of X^*. Identifying X with a subspace of Y^* suppose that there are weak*-compact subsets $(K_j)_{j=1}^\infty$ of Y^* such that $Y^* \backslash X = \bigcup_{j=1}^\infty K_j$ and* $\text{dist}(K_j, X) > \varepsilon_j > 0$ *for $j \in \mathbf{N}$.*

Then there is an isometric embedding $i : X \mapsto C(\Delta)$ whose adjoint $i^* : \mathscr{M}(\Delta) \mapsto X^*$ induces a quotient map, denoted q, from $L^1(\mu)$ onto Y such that the embedding $q^* : Y^* \mapsto L^\infty(\Delta, \mu)$ verifies $\text{dist}(q^*(K_j), C(\Delta)) \geq \varepsilon_j/2$ *for $j \in \mathbf{N}$.*

Proof. Let $\mathscr{D} = (y_n)_{n=0}^\infty$ be norm-dense in the unit ball of Y, $y_0 = 0$ and fix a metric d on the unit ball of X^* that induces the $\sigma(X^*, X)$-topology. Also fix an element s_0 of Δ.

Note first that for $y^* \in Y^*$ with $\text{dist}(y^*, X) > \varepsilon > 0$ and $\delta > 0$ we may find $y_n \in \mathscr{D}$ such that $d(y_n, 0) < \delta$ and $\langle y_n, y^* \rangle > \varepsilon$. Indeed, we may find a finite dimensional subspace $E \subseteq X$ such that

$$E^{\perp} = \{y \in Y : y|_E = 0\} \subseteq \{y \in Y : d(y,0) < \delta\}.$$

As the image of y^* in the quotient Y^*/E has norm larger than ε, we may find $y \in E^{\perp}$, $\|y\| \le 1$, such that $\langle y, y^* \rangle > \varepsilon$. Finally approximate y by some $y_n \in \mathscr{D}$ to get the desired conclusion.

We shall now inductively construct an increasing sequence $(i_k)_{k=1}^{\infty}$ in $\mathbf{N}$ and maps $\varphi_k : \Delta_{i_k} \mapsto \mathscr{D}$ such that

(i) for $k \ge 2$, $d(\varphi_k(t), \varphi_k(t')) < 2^{-k+1}$ for all $t, t' \in \Delta_{i_k}$ verifying $\pi_{i_k, i_{k-1}}(t) = \pi_{i_k, i_{k-1}}(t')$.

(ii) for $k \ge 1$, $\varphi_k(\Delta_{i_k})$ contains a $2^{-(k+1)}$-net of $(\mathrm{ball}(X^*), d)$ as well as $(y_n)_{n \le k}$,

(iii) for $k \ge 2$ and $s \in \Delta_{i_{k-1}}$

$$\mu_{i_k}\{t \in \Delta_{i_k} : \pi_{i_k, i_{k-1}}(t) = s \text{ and } \varphi_k(t) = \varphi_{k-1}(s)\}$$
$$> (1 - 2^{-k})\mu_{i_k}\{t \in \Delta_{i_k} : \pi_{i_k, i_{k-1}}(t) = s\} = (1 - 2^{-k})2^{-i_{k-1}}.$$

(iv) for $k \ge 2$, denoting $s_{k-1} = \pi_{i_{k-1}}(s_0)$, we have $\varphi_{k-1}(s_{k-1}) = y_0 = 0$ and for $1 \le j \le k-1$ and $y^* \in K_j$ there is $t \in \Delta_{i_k}, \pi_{i_k, i_{k-1}}(t) = s_{k-1}$ such that

$$\langle \varphi_k(t), y^* \rangle > \varepsilon_j.$$

To start the induction, find N large enough such that $(y_n)_{n \le N}$ forms a $\frac{1}{4}$-net of $(\mathrm{ball}(X^*), d)$ and let $i_1 \in \mathbf{N}$ be large enough such that there is a surjection φ_1 from Δ_{i_1} onto $(y_n)_{n \le N}$.

If i_{k-1} and φ_{k-1} are defined, fix $1 \le j \le k-1$ and $y^* \in K_j$. By the observation above there is $y_n \in \mathscr{D}$ such that

$$d(y_n, 0) < 2^{-k} \tag{1}$$

and

$$\langle y_n, y^* \rangle > \varepsilon_j. \tag{2}$$

Note that inequality (2) holds true for all $z^* \in Y^*$ in a $\sigma(Y^*, Y)$-neighborhood of y^*. By compactness we may find finitely many $(n_q^k)_{q=1}^{p_k}$ such that (1) holds true for $y_{n_q^k}$, $1 \le q \le p_k$, and, for $1 \le j \le k-1$ and $y^* \in K_j$,

$$\max_{1 \le q \le p_k} \langle y_{n_q^k}, y^* \rangle > \varepsilon_j. \tag{3}$$

We shall first define a map ψ_k on the atom $A_{s_{k-1}} = \{t \in \Delta : \pi_{i_{k-1}}(t) = s_{k-1}\}$ of Δ. Choose $\psi_k : A_{s_{k-1}} \mapsto \mathscr{D}$ in such a way that

–there is $r_k > i_{k-1}$ such that $\psi_k | A_{s_{k-1}}$ depends only on the first r_k coordinates of Δ,

- $\mu\{t \in A_{s_{k-1}}: \psi_k(t) = 0\} > (1 - 2^{-k}) \cdot \mu(A_{s_0})$,
- $\pi_{r_k}(s_0) = \pi_{r_k}(t)$ implies that $\psi_k(t) = 0$,
- $\psi_k(A_{s_{k-1}}) \subseteq \{y \in \mathscr{D}: d(y, 0) < 2^{-k}\}$,
- $\psi_k(A_{s_{k-1}})$ contains a $2^{-(k+1)}$-net of $\{y \in \mathscr{D}: d(y, 0) < 2^{-k}\}$, those elements of $(y_n)_{n \leq k}$ such that $d(y_n, 0) < 2^{-k}$ as well as $(y_{n_q^k})_{q=1}^{p_k}$.

Next, for $s \in \Delta_{i_{k-1}}$, $s \neq s_{k-1}$, define ψ_i on the atom $A_s = \{t \in \Delta: \pi_{i_{k-1}}(t) = s\}$ such that

- $\psi_k | A_s$ depends only on finitely many coordinates of Δ,
- $\mu\{t \in A_s: \psi_k(t) = \varphi_{k-1}(s)\} > (1 - 2^{-k})\mu(A_s)$,
- $\psi_k(A_s) \subseteq \{y \in \mathscr{D}: d(y, \varphi_{k-1}(s)) < 2^{-k}\}$,
- $\psi_k(A_s)$ contains a $2^{-(k+1)}$-net of $\{y \in \mathscr{D}: d(y, \varphi_{k-1}(s)) < 2^{-k}\}$ as well as those elements of $(y_n)_{n<k}$ verifying $d(y_n, \varphi_{k-1}(s)) < 2^{-k}$.

Finally define ψ_k on all of Δ by its values on the atoms $(A_s)_{s \in \Delta_{i_{k-1}}}$ and let $i_k \geq r_k$ be large enough so that ψ_k depends only on the coordinates $\{1, \ldots, i_k\}$ of Δ, i.e. $\psi_k = \varphi_k \circ \pi_{i_k}$ for a map $\varphi_k: \Delta_{i_k} \mapsto \mathscr{D}$. One quickly verifies that y_k satisfies (i)–(iv) above. This completes the inductive step.

Considering again $\psi_k = \varphi_k \circ \pi_{i_k}$ note that by (i) ψ_k converges uniformly towards a continuous function

$$\psi: \Delta \mapsto (\text{ball}(X^*), d).$$

For $n_0 \in \mathbf{N}$ we infer from (ii) that y_{n_0} is in the range of φ_{n_0} and we obtain from (iii) that

$$\mu\{\psi^{-1}(y_{n_0})\} > \left[\prod_{n>n_0} (1 - 2^{-n})\right] \cdot 2^{-i_{n_0}} > 0. \tag{4}$$

Whence, for every $n \in \mathbf{N}$, ψ takes the value y_n on a set of positive μ-measure. In particular $\text{range}(\psi) \supseteq \mathscr{D}$ and, as it is w^*-compact, we have

$$\text{range}(\psi) = \text{ball}(X^*). \tag{5}$$

Defining $i: X \mapsto C(\Delta)$ by $x \mapsto x \circ \psi$, we obtain an isometric embedding in view of (5). Denote by q the restriction of the adjoint map i^* to $L^1(\Delta, \mu)$. From (4) we obtain that

$$q(\text{ball}(L^1(\mu))) \supseteq \mathscr{D}. \tag{6}$$

On the other hand

$$q(\text{ball}(L^1(\mu))) \subseteq \text{ball}(Y). \tag{7}$$

Indeed, from (iii) we infer again that, for $n_0 \in \mathbf{N}$

$$\mu\{\psi^{-1}(\text{range}(\varphi_{n_0}))\} > \prod_{n>n_0} (1 - 2^{-n}).$$

As range(φ_{n_0}) is a (finite) subset of $\mathscr{D}$ and the right-hand side above tends to 1 when $n_0 \to \infty$ we conclude that

$$\mu\{\psi^{-1}(\mathscr{D})\} = 1, \tag{8}$$

which readily implies (7). Combining (6) and (7) we have shown that q is a quotient map from $L^1(\mu)$ onto Y and therefore $q^*: Y^* \mapsto L^\infty(\mu)$ is an isometric embedding. We shall now show that, for $j \in \mathbf{N}$ and $y^* \in K_j$

$$\operatorname{dist}(q^*(y^*), C(\Delta)) \geq \varepsilon_j/2. \tag{9}$$

First note that it follows from (8) that $q^*(y^*) = y^* \circ \psi$ in the sense that the function $y^* \circ \psi$ on Δ is a representative of the element $q^*(y^*)$ of $L^\infty(\Delta, \mu)$. To show (9) it will suffice to prove that, for every neighborhood V of s_0, the function $y^* \circ \psi$ takes values larger than ε_j and others less than or equal to 0 on subsets of V of strictly positive μ-measure.

Note that V contains, for some $k \in \mathbf{N}$, an atom

$$A_{s_{k-1}} = \{t \in \Delta: \pi_{i_{k-1}}(t) = s_{k-1}\}.$$

By (iv) and once again (iii) we may find subsets B_1 and B_2 of $A_{s_{k-1}}$ of strictly positive μ-measure such that

$$t \in B_1 \Rightarrow \langle \psi(t), y^* \rangle = 0, \qquad t \in B_2 \Rightarrow \langle \psi(t), y^* \rangle > \varepsilon_j.$$

This proves (9) and therefore the proposition.

Remark I.6. Note that in the construction above, we get that for each y^* in K_j, the function $y^* \circ \varphi$ has a discontinuity at any point $s_0 \in \Delta$ which has been initially fixed. Also note that, with a little more care in the above construction, one may replace the conclusion $\operatorname{dist}(q^*(K_j), C(\Delta)) \geq \varepsilon_j/2$ by $\operatorname{dist}(q^*(K_j), C(\Delta)) > \varepsilon$.

Theorem I.7. *Let X be a separable Banach space. Then*:

(a) *X has the P.C.P if and only if it has a JT-type decomposition.*

(b) *X has the R.N.P if and only if it has a JM-type decomposition.*

Proof. We first show that the existence of a JT-type (resp. JM-type) decomposition is a sufficient condition for X to have the P.C.P (resp. the R.N.P). In the first case this follows directly from [BR1] as a JT-type decomposition is in particular a boundedly complete skipped blocking finite dimensional decomposition.

In the second case this also follows easily from the characterization of R.N.P spaces in terms of $w^* - H_\delta$-sets [GM3, Theorem I.8]: If X has a JM-type decomposition, let Y be the space spanned by $([X_m]^{\perp}_{m \neq n})_{n=1}^{\infty}$ in X^*. By Lemma I.15 below, the restriction mapping defines an isomorphic embedding $j: X \mapsto Y^*$. An element $y^* \in Y^*$ is in $Y^* \backslash j(X)$ iff there is $i \in \mathbf{N}$, $\alpha > 0$ and

$m \in \mathbf{N}$ such that $\langle y^*, y_n^i \rangle \geq \alpha$ for $n \geq m$. Hence we may write $Y^* \backslash j(X)$ as a countable union of w^*-compact convex sets:

$$Y^* \backslash j(X) = \bigcup_{i \in N} \bigcup_{\alpha > 0} \bigcup_{m \in \mathbf{N}} \bigcap_{n \geq m} \{y^* \in Y^* : \langle y^*, y_n^i \rangle \geq \alpha\}.$$

So $j(X)$ is then a $w^* - H_\delta$-subset of Y^* and therefore X has R.N.P. [GM3, Theorem I.8].

Suppose now conversely that X has P.C.P (resp. R.N.P). We do the proof simultaneously for the two cases since most of the construction will be identical. By [GM3, Theorem IV.8] we may find a separable norming subspace Y of X^* and $\sigma(Y^*, Y)$ compact (resp. convex w^*-compact in the case of R.N.P) subsets $(K_j)_{j=1}^\infty$ of Y^* such that

(i) $Y^* \backslash X = \bigcup_{j=1}^\infty K_j$;
(ii) $\operatorname{dist}(K_j, X) \geq \varepsilon_j > 0$.

Here we identify X with an isometric subspace of Y^*.

By Proposition I.5 we may find an isometric embedding j of X into a space E with a monotone basis $(e_i^*)_{i=1}^\infty$ such that j^* induces a quotient map q from B onto Y where B is the norm closed linear span of the biorthogonal functionals $(e_i)_i$ in E^* in such a way that $\operatorname{dist}(q^*(K_j), E) \geq \varepsilon_j/2$.

We identify X, E and Y^* isometrically with subspaces of B^*. The inclusions are given by the diagram:

$$\begin{array}{ccc} Y^* & \xrightarrow{q^*} & B^* \\ \uparrow & & \uparrow \\ X & \xrightarrow[j]{} & E \end{array}$$

On B^* we define for $n \in \mathbf{N}$, the contractive projections $\pi[1, n] : B^* \mapsto B^*$ by $\pi[1, n](z^*) = \sum_{i=1}^n z^*(e_i) e_i^*$. For $n \leq m$ denote $\pi[n, m] = \pi[1, m] - \pi[1, n[$ and $\pi[n, \infty[= \mathrm{Id} - \pi[1, n[$.

We shall first prove the following

Claim. Given j and $n \in \mathbf{N}$ there is $m \geq n$ such that, for every $y^* \in K_j$, $\|\pi[n, m](y^*)\| > \varepsilon_j/4$.

Indeed, for every $y_0^* \in K_j$, the distance of y_0^* to E is larger than $\varepsilon_j/2$, therefore $\|\pi[n, \infty[(y_0^*)\| > \varepsilon_j/3$. Hence we may find $m(y_0^*) \geq n$ such that $\|\pi[n, m](y_0^*)\| > \varepsilon_j/4$. Note now that $V_{y_0^*} = \{y^* \in K_j : \|\pi[n, m](y^*)\| > \varepsilon_j/4\}$ defines a relative weak-star neighborhood of y_0^* in K_j. By compactness and the monotonicity of the basis $(e_i^*)_{i=1}^\infty$ we may find $m \geq n$ such that $\|\pi[n, m](y^*)\| > \varepsilon_j/4$ for all $y^* \in K_j$. The claim is proved.

We now start the inductive construction: Let $(v_k)_{k=1}^\infty$ be a dense sequence in X. Choose X_1 to be a finite dimensional subspace of X containing v_1. Find

$n_1 \in \mathbf{N}$ such that $\|\pi[n_1, \infty[(x)\| < 2^{-1}\|x\|$ for $x \in X_1$, and choose $q_1 \geq n_1$ such that $\operatorname{dist}(\pi[n_1, q_1](K_1), \{0\}) > \varepsilon_1/4$.

For the inductive step suppose we have defined finite dimensional spaces $(X_i)_{i \leq k}$ and integers $n_1 \leq q_1 < n_2 \leq q_2 < \cdots < n_k \leq q_k$ such that

(i) $\|\pi[n_i, \infty[(x)\| < 2^{-i}\|x\|$ if $x \in [X_j]_{j \leq i}$, $1 \leq i \leq k$,
(ii) $\pi[1, q_{i-2}](x_i) = 0$ if $x_i \in X_i$, $3 \leq i \leq k$,
(iii) $X_i \cap ([X_j]_{j<i}) = \{0\}$ for $2 \leq i \leq k$,
(iv) $\pi[1, q_{i-1}](X) = \pi[1, q_{i-1}]([X]_{j \leq i})$ for $2 \leq i \leq k$,
(v) $\operatorname{dist}(\pi[n_i, q_i](K_j), \{0\}) > \varepsilon_j/4$ for $1 \leq j \leq i \leq k$,
(vi) $v_i \in [X_j]_{j \leq i}$ for $1 \leq i \leq k$.

Using elementary linear algebra, we may find a finite-dimensional subspace X_{k+1} such that (ii), (iii) (iv) and (vi) are satisfied with k replaced by $k+1$. Indeed, we may find finitely many elements $\{x_1^{k+1}, \ldots, x_{p_{k+1}}^{k+1}\}$ in X such that if we denote $X_{k+1} = \operatorname{span}\{x_1^{k+1}, \ldots, x_{p_{k+1}}^{k+1}\}$, then $\pi[1, q_{k-1}](X_{k+1}) = \{0\}$, $[X_j]_{j \leq k} \cap X_{k+1} = \{0\}$ and $\pi[1, q_k]([X_j]_{j \leq k+1}) = \pi[1, q_k](X)$.

Clearly we may also assume in addition that $v_{k+1} \in [X_j]_{j \leq k+1}$. Now find $n_{k+1} \in \mathbf{N}$, $n_{k+1} > q_k$, such that (i) is satisfied and apply the above claim to find $q_{k+1} \in \mathbf{N}$, $q_{k+1} \geq n_{k+1}$, such that (v) holds true (with k replaced by $k+1$). This completes the inductive step.

To find the functionals $(y_k^j)_{j=1}^{m_k}$ required in Definition I.1 we now distinguish the cases of R.N.P and P.C.P: If X has R.N.P we apply (v) and the Hahn-Banach theorem to find, for $k, j \in \mathbf{N}$, $1 \leq j \leq k$, elements z_k^j in the unit ball of B such that $z_k^j \in [e_i]_{i=n_k}^{q_k}$ and

$$\inf_{y^* \in K_j} \langle y^*, z_k^j \rangle > \varepsilon_j/4.$$

Denote by w_k^j the restriction of z_k^j to Y^*; clearly $w_k^j \in Y$, $\|w_k^j\| \leq 1$. Note that because of (i) and (ii) the restriction of w_k^j to $[X_i]_{i \neq k+1}$ has norm less than 2^{-k}. Hence we may find $y_k^j \in Y$, $\|y_k^j\| = 1$, $y_k^j \in [X_i]_{i \neq k}^{\perp}$, $\|w_k^j - y_k^j\| < 2^{-k+1}$ such that

$$\inf_{y^* \in K_j} \langle y^*, y_k^j \rangle > (\varepsilon_j/4 - \|K_j\| \cdot 2^{-k})/(1 + 2^{-k}).$$

In the case of P.C.P we proceed similarly but, instead of the Hahn-Banach theorem, we apply a compactness argument to find finitely many $(z_k^i)_{i=1}^{m_k}$ in the ball of B supported by $[e_p]_{p=n_k}^{q_k}$ such that

$$\inf_{y^* \in K_j} \max_{1 \leq i \leq m_k} \langle y^*, z_k^i \rangle > \varepsilon_j/4, \qquad 1 \leq j \leq k.$$

Note that, contrary to the case of R.N.P, we have no control over the growth of m_k. Using the same perturbation argument as above we may find $(y_k^i)_{i=1}^{m_k}$ in the unit sphere of Y and orthogonal to $[X_j]_{j \neq k}$ such that

$$\inf_{y^* \in K_j} \max_{1 \leq i \leq m_k} \langle y^*, y_k^i \rangle > (\varepsilon_j/4 - \|K_j\| \cdot 2^{-k})/(1 + 2^{-k}).$$

The construction is completed.

Let us verify that the conditions of a JT-type decomposition are satisfied: Given $x^{**} \in X^{**}$ and $x_n = p_n^{**}(x^{**})$ where $(p_n)_{n=1}^{\infty}$ are the projections on $(X_n)_{n=1}^{\infty}$ with kernels $[X_m]_{m\neq n}$ verifying $p_n p_m = 0$ if $n \neq m$ and $p_n^* y_n^i = y_n^i$ for $1 \leq i \leq m_n$. Let $x \in Y^*$ be the restriction of x^{**} to Y. Clearly $\|x\| \leq \|x^{**}\|$.

If $x \in Y^* \backslash X$ then $x \in K_j$ for some $j \in \mathbf{N}$, i.e., in the case of P.C.P,

$$\max_{1\leq i\leq m_k} \langle x, y_k^i\rangle > (\varepsilon_j/4 - \|K_j\| \cdot 2^{-k})/(1+2^{-k}) \quad \text{for all } k \in \mathbf{N},$$

and in the case of R.N.P

$$\langle x, y_k^j\rangle > (\varepsilon_j/4 - \|K_j\| \cdot 2^{-k})/(1+2^{-k}) \quad \text{for all } k \in \mathbf{N}.$$

In other words if condition (1) (resp. (2)) of Definition I.1 holds true, then $x \in X$. Clearly x has the proper coordinates, i.e. $p_k(x) = x_k$ for $k \in \mathbf{N}$. We have proved that (a)(ii) (resp. (b)(ii)) of Definition I.1 is satisfied.

We still have to verify that $(X_n)_{n=1}^{\infty}$ is a boundedly complete skipped blocking decomposition.

Clearly by (vi) the span of $(X_n)_{n=1}^{\infty}$ is dense in X. Let $(c_j)_{j=1}^{\infty}$ and $(d_j)_{j=1}^{\infty}$ be positive integers such that $c_j < d_j + 1 < c_{j+1}$ and $z_j \in [X_i]_{i=c_j}^{d_j}$ such that the partial sums $s_j = z_1 + \cdots + z_j$ stay bounded. Note first that $(z_j)_{j=1}^{\infty}$ is a basic sequence. This easily follows form (i) and (ii) as z_j and z_{j+1} are essentially (i.e., up to an error, controlled by (i)) contained in the subspaces of E spanned by the disjoint subsets $(e_i^*)_{i=1}^{n_{d_j}-1}$ and $(e_i^*)_{i=q_{(c_{j+1}-2)}+1}^{\infty}$ respectively. Note that $c_{j+1} - 2 \geq d_j$ hence $q_{(c_{j+1}-2)} \geq n_{d_j}$. More explicitly, there are $u_j \in X$, $u_j \in \operatorname{span}((e_i^*)_{i=q_{c_j-2}+1}^{n_{d_j}-1})$, such that $\|u_j - z_j\| < 2^{-d_j}\|z_j\|$.

Similarly as above we conclude that the $\sigma(Y^*, Y)$-limit s of s_j lies in fact in X as, for every $j \in \mathbf{N}$ and $1 \leq i \leq m_{n_{d_j}}$, $\langle s, y_{n_{d_j}}^i\rangle = 0$. Hence $\sum_{j=1}^{\infty} z_j$ converges in norm to s. This shows that $(X_n)_{n=1}^{\infty}$ is a boundedly complete skipped blocking decomposition. □

We now reformulate the above characterizations in terms of semiembeddings into c_0.

Definition I.8. A one-to-one continuous linear operator $S\colon X \mapsto c_0(I)$ is called

(a) a *JT-type injection* if the index set $I \subseteq \mathbf{N}\times\mathbf{N}$ has a countable partition $(I_n)_{n=1}^{\infty}$ with $I_n = \{(n,1), (n,2), \ldots, (n,k_n)\}$ such that, whenever $x^{**} \in X^{**}$, the condition

$$\liminf_{n\mapsto\infty} \|\pi_n^{**} S^{**}(x^{**})\| = 0 \tag{3}$$

implies that there is an $x \in X$, $\|x\| \leq \|x^{**}\|$, with $S(x) = S^{**}(x^{**})$.

Here $\pi_n\colon c_0(I) \mapsto c_0(I_n)$ denotes the canonical projection.

(b) a *JM-type* injection if S is a JT-type injection and in addition (3) may be replaced by

(4) $$\liminf_{n\to\infty} S^{**}(x^{**})(n,j) \le 0 \quad \text{and} \quad \liminf_{n\to\infty}(-S^{**}(x^{**})(n,j)) \le 0$$

for all $j \in \mathbf{N}$.

Remark I.9. The archexamples of the above situation are the natural injections $S_{J_*T}: J_*T \mapsto c_0(T)$ and $S_{J_*M}: J_*M \mapsto c_0(\mathbf{N}\times\mathbf{N})$ where we may choose I_n as a reordering of $\{t: |t| = n\}$ in the first case and of $\{(n,j) \in \mathbf{N}\times\mathbf{N}: 1 \le j \le n\}$ in the second case.

Note that a JT-type injection (and therefore a JM-type injection) is necessarily a semiembedding. Indeed, the closure of $S(\mathrm{ball}(X))$ equals the intersection of $S^{**}(\mathrm{ball}(X^{**}))$ with c_0. If $\|x^{**}\| \le 1$ is such that $S^{**}(x^{**}) \in c_0$ it follows in particular from (3) that $S^{**}(x^{**})$ belongs to $S(\mathrm{ball}(X))$ which is therefore closed in c_0.

Theorem I.10. *Let X be a separable Banach space. Then*:

(a) *X has the P.C.P if and only if there is a JT-type injection into c_0.*

(b) *X has the R.N.P if and only if there is a JM-type injection into c_0.*

Proof. Again the results of [GM1] and [GM2] easily imply that the existence of a JT-type (resp. JM-type) injection is a sufficient condition for P.C.P (resp. R.N.P): Find strictly positive scalars $(\lambda_t)_{t\in I}$ such that $\sum_{t\in I} |\lambda_t|^2 < \infty$ and let

$$D: c_0(I) \mapsto l^2(I)$$
$$(x_t)_{t\in I} \mapsto (\lambda_t x_t)_{t\in I}$$

be the diagonal map with weights λ_t. The maps D as well as DS are compact and one-to-one. Note that the norm closure of $DS(\mathrm{ball}(X))$ equals $D^{**}S^{**}(\mathrm{ball}(X^{**}))$. In the case of P.C.P we may write

$$\overline{DS(\mathrm{ball}(X))}\backslash DS(\mathrm{ball}(X)) = \left\{ \begin{array}{l} D^{**}S^{**}x^{**}; \|x^{**}\| \le 1 \text{ and } \exists N \in \mathbf{N}, \alpha > 0, \\ \text{s.t. for all } n \ge N, \|\pi_n^{**}S^{**}(x^{**})\| \ge \alpha \end{array} \right\}$$

which one may write as a countable union of compact subsets of $l^2(I)$ and in the case of R.N.P

$$\overline{DS(\mathrm{ball}(X))}\backslash DS(\mathrm{ball}(X)) = \left\{ \begin{array}{l} D^{**}S^{**}x^{**}: \|x^{**}\| \le 1 \text{ and there is } m \in \mathbf{N}, \\ j \in \mathbf{N}, \alpha > 0 \text{ and } \varepsilon = \pm 1 \text{ s.t. } \forall n \ge m \\ \varepsilon D^{**}S^{**}x^{**}(n,j) \ge \lambda_{(n,j)}\alpha. \end{array} \right\}$$

which one may write as a countable union of compact convex subsets of $l^2(I)$. Hence DS is a compact G_δ (resp. H_δ) embedding and we are done by [GM1, Theorems II.1 and III.1].

Conversely suppose X has P.C.P (resp. R.N.P) and apply I.7 to find a JT-type (resp. JM-type) decomposition $(X_n)_{n=1}^{\infty}$ of X. Let $(y_n^i)_{i=1}^{m_n}$ be as in Definition I.1 and find supersets $(y_n^i)_{i=1}^{k_n}$ of $(y_n^i)_{i=1}^{m_n}$ where $k_n \ge m_n$ $\|y_n^i\| = 1$,

$y_n^i \in [X_m]_{m\neq n}^{\perp}$ and such that $(y_n^i))_{i=1}^{k_n}$ separate the points of X_n. It follows from Remark I.16 below that the elements $((y_n^i)_{i=1}^{k_n})_{n=1}^{\infty}$ separate the points of X.

Let $I = \bigcup_{n=1}^{\infty}((n, i)_{i=1}^{k_n})_{n=1}^{\infty}$ and define the bounded linear operator:

$$S\colon X \mapsto c_0(I)$$
$$x \mapsto ((\langle x, y_n^i\rangle_{i=1}^{k_n})_{n=1}^{\infty}.$$

It is easy to check that S is a JT (resp. JM) injection. □

Remark I.11. One might ask which class of Banach spaces we obtain if we assume the dimensions $(\dim(X_n))_{n=1}^{\infty}$ in Definition I.1 of JT-type decompositions to be uniformly bounded. We shall make this concept precise in Definition I.12. However, this class turns out to be too restrictive to characterize R.N.P. (see Theorem I.14).

Definition I.12. A sequence $(X_n)_{n=1}^{\infty}$ of finite dimensional subspaces of a Banach space X is called a *J-type decomposition* if it is a JT-type decomposition and in addition $(\dim(X_n))_{n=1}^{\infty}$ is bounded.

Remark I.13. The archexample of a Banach space X admitting a J-type decomposition (into 1-dimensional spaces) is J_*, the predual of James space J.

Theorem I.14. *If a Banach X has a J-type decomposition then it is a isomorphic to a dual Banach space.*

Proof. Let Y be the closed linear span of $([X_m]_{m\neq n}^{\perp})_{n=1}^{\infty}$ in X^* and let $j\colon X \mapsto Y^*$ be the restriction map. Admitting the subsequent Lemma I.15, the map j is an isomorphic embedding. We shall show that $j(X)$ has finite codimension in Y^*. This will finish the proof as it is easy to show that such a subspace is isomorphic to a dual space.

For each $n \in \mathbf{N}$, denote by P_n the quotient map $P_n\colon X \to X/[X_m]_{m\neq n}$.

We may identify $(X/_{[X_m]_{m\neq n}})^*$ isometrically with $[X_m]_{m\neq n}^{\perp}$. Note that P_n^* takes its values in Y whence P_n^{**} factors through Y^*, thus defining maps on Y^* which we still denote by $P_n\colon Y^* \mapsto X/_{[X_m]_{m\neq n}}$.

Let $((y_n^i)_{i=1}^{m_n})_{n=1}^{\infty}$ be as in Definition I.1. Note that the biorthogonality condition $y_n^i \in [X_m]_{m\neq n}^{\perp}$ implies that $y_n^i = y_n^i \circ P_n$. Hence $y^* \in Y^*$ belongs to X if (and only if) $\liminf_{n\mapsto\infty} \|P_n y^*\| = 0$.

Let $d \in \mathbf{N}$ such that $\dim(X_n) = d$ for all n in an infinite subset $A \subseteq \mathbf{N}$ (we shall in fact only use the hypothesis that $\liminf_{n\to\infty}(\dim(X_n)) < \infty$). For $n \in \mathbf{N}$ fix an isomorphism

$$i_n\colon X/_{[X_m]_{m\neq n}} \mapsto l_d^2$$

such that $\|i_n\| = 1$ and $(\|i_n^{-1}\|)_{n=1}^{\infty}$ is bounded by some constant M (by F. John's lemma we may choose the isomorphisms such that $M \le d^{1/2}$).

We shall show that $\dim(Y^*/j(X)) \le d$. Suppose to the contrary that $y_1^*, \dots, y_{d+1}^*$ are $d+1$ elements of Y^* such that the images of $(y_j^*)_{j=1}^{d+1}$ in $Y^*/j(X)$ are linearly independent.

Let $\mathscr{U}$ be a nontrivial ultrafilter on the set A and define

$$w_j = \lim_{n\in\mathscr{U}} i_n P_n(y_j^*).$$

Hence $(w_j)_{j=1}^{d+1}$ are $d+1$ vectors in l_d^2 and we may find a nontrivial representation of 0, i.e. $0 = \sum_{j=1}^{d+1} \lambda_j w_j$. Let $y_0^* = \sum_{j=1}^{d+1} \lambda_j y_j^*$. Then

$$\liminf_{n\to\infty} \|P_n(y_0^*)\| \le M \cdot \lim_{n\in\mathscr{U}} \left\| i_n P_n \left(\sum_{j=1}^{d+1} \lambda_j y_j^* \right) \right\| = 0.$$

Whence $y_0^* \in j(X)$, contrary to the above assumption.

We have shown that $\dim(Y^*/j(X)) \le d$, thus finishing the proof. □

Lemma I.15. *Let $(X_n)_{n=1}^\infty$ be a skipped blocking finite dimensional decomposition of a Banach space X. Then the space Y spanned by $([X_m]_{m\ne n}^\perp)_{n=1}^\infty$ in X^* c-norms X for some $c > 0$, i.e.*

$$c\|x\| \le \sup\{\langle x, y\rangle : y \in Y, \|y\| \le 1\}.$$

Hence the restriction map $j: X \mapsto Y^*$ is an isomorphic embedding.

Proof. We shall first show that if $(X_n)_{n=1}^\infty$ is a skipped blocking decomposition then there is $c > 0$ such that the natural projections $Q[1, n]: [X_m]_{m\ne n+1} \mapsto [X_m]_{m\le n}$ verify $\|Q[1, n]\| \le c^{-1}$. For that, suppose that

$$\limsup_{n\to\infty} \|Q[1, n]\| = +\infty.$$

We shall construct by induction a strictly increasing sequence $(n_k)_{k=0}^\infty$ such that if $Z_k = \mathrm{span}\{X_n\}_{n=n_k+2}^{n_{k+1}}$ the projections from $Z = \mathrm{span}\{Z_k\}_{k\ge 0}$ onto $\mathrm{span}\{Z_k\}_{k\le n}$ fail to be uniformly bounded, which contradicts the assumption that $(X_n)_n$ is a skipped blocking decomposition.

Start the inductive construction by taking $u_0 = 0$, $m_0 = 2$ and suppose that $n_0 < \cdots < n_k$ and $m_k > n_k + 1$ are defined.

Then for $n > m_k$, the canonical projections

$$R_{[n,1]}: [X_m]_{m\ne n+1, m\notin\{n_0,\dots,n_k\}} \mapsto [X_m]_{m\le n, m\notin\{n_0,\dots,n_k\}}$$

remain unbounded too.

Hence we may find $n_{k+1} > m_k$ and $m_{k+1} > n_{k+1} + 1$ such that

$$R_{[n_{k+1},1]}: [X_m]_{m\ne n_{k+1}, m\le m_{k+1}, m\notin\{n_0,\dots,n_k\}} \mapsto [X_m]_{m\le n_{k+1}, m\notin\{n_0,\dots,n_k\}}$$

satisfies

$$\|R_{[n_{k+1},1]}\| > k.$$

This completes the inductive step.

Now let $x_0 \in X$, $\|x_0\| = 1$, and find for $\varepsilon > 0$, $n \in \mathbf{N}$ and $x_0' \in [X_m]_{m \le n}$, $\|x_0'\| = 1$ such that $\|x_0 - x_0'\| < \varepsilon$. Note that, for every $z \in [X_m]_{m \ge n+2}$, $\|x_0' + z\| \ge c$ whence, denoting $P[1, n+1]\colon X \mapsto X/[X_m]_{m \ge n+2}$ the canonical quotient map, we obtain $\|P[1, n+1](x_0')\| \ge c$ and $\|P[1, n+1](x_0)\| > c - \varepsilon$.

Noting that $P[1, n+1]^*$ defines an isometric embedding of $(X/[X_m]_{m \ge n+2})^*$ into $[X_m]^{\perp}_{m \ge n+2}$ (which equals the span of $([X_m]^{\perp}_{m \ne j})_{j=1}^{n+1}$) we may find $y \in [X_m]^{\perp}_{m \ge n+2}$, $\|y\| = 1$, such that $\langle x_0, y\rangle > c - \varepsilon$. This clearly finishes the proof. □

Remark I.16. It follows from Lemma I.15 and under the same hypothesis that the quotient maps $P_n\colon X \mapsto X/[X_m]_{m \ne n}$ separate the points of X. As the "coordinate projections" $p_n\colon X \mapsto X_n$ have the same kernel as the quotient maps P_n, namely $[X_m]_{m \ne n}$, we conclude that the coordinate projections $(p_n)_{n=1}^{\infty}$ also separate the points of X.

Remark I.17. In view of Theorem I.14, one may ask whether there exists a "growth condition" on the dimension of the factors in a JT-type decomposition which insures that the space has the R.N.P. Note that the James-matrix space defined above may suggest that the condition $\liminf_{n \to \infty}(\dim(X_n)/n) < \infty$ is sufficient. Unfortunately this is not the case. Indeed, for each increasing sequence of integers $(n_k)_k$ tending to $+\infty$, one can construct a Banach space X failing the R.N.P but having a JT-type decomposition $(X_k)_k$ such that $\dim(X_k) = n_k$ for each $k \in \mathbf{N}$.

II. Boundedly complete and l^1-skipped blocking infinite dimensional decompositions

We start this section by establishing the following stability result for "strong regularity".

Theorem II.1. *If a Banach space X admits a complemented boundedly complete skipped blocking decomposition into strongly regular subspaces $(X_n)_n$ then X is strongly regular.*

The proof will follow from a more general principle that we emphasize since it might be of independent interest: Let us say that a property (P) on Banach spaces is *stable under G_δ-embeddings* (resp. *stable under l_2-sums*) if whenever $T\colon X \mapsto Y$ is a G_δ-embedding and Y has (P) (resp. if $X = (\sum_n \bigoplus X_n)_{l_2}$ and each X_n has (P)) then X has the property (P). Similarly, we can define stability under boundedly complete skipped blocking decomposition.

We have the following.

Proposition II.2. *If (P) is a property on Banach spaces that is stable under G_δ-embeddings and l_2-sums, then (P) is stable under the formation of complemented boundedly complete skipped blocking decompositions.*

The proof of the above proposition follows immediately from the following fact established in [GM2, Proposition II.4]. If $(X_n)_{n=1}^{\infty}$ is a complemented boundedly complete skipped blocking decomposition for a Banach space X, then there exists a G_δ-embedding from X into $(\sum_{n=1}^{\infty} \oplus X_n)l_2$.

Theorem II.1 follows then from Proposition II.3 and II.4 below.

Proposition II.3. *Strong regularity is stable under* l_2*-sums.*

Proof. Recall that X is strongly regular iff for every closed, convex, bounded $C \subseteq X$ the set of S.C.S-points of C is norm-dense in C [GGMS, III.6]. Recall that a point $x \in C$ is an S.C.S-point if it is contained in convex combinations of slices of arbitrarily small diameter, i.e. for $\varepsilon > 0$, there are slices $S_1, \dots, S_n$ of C such that $x \in n^{-1}(S_1 + \cdots + S_n)$ and $\mathrm{diam}(n^{-1}(S_1 + \cdots + S_n)) < \varepsilon$.

Let $(X_n)_{n=1}^{\infty}$ be a sequence of strongly regular spaces, $X = (\sum_{n=1}^{\infty} \oplus X_n)_2$ and C closed and convex in X with $\|C\| = \sup\{\|x\| : x \in C\} = 1$. Denote $C_n = \pi[1, n](C)$, where $\pi[1, n]$ denotes the natural projection of X onto $(\sum_{i=1}^{n} \oplus X_i)_2$.

One easily checks that a finite product of strongly regular spaces is strongly regular. Hence we may find an $n_0 \in \mathbf{N}$ and an S.C.S-point z_0 of $\overline{C}_{n_0}$ such that $\|z_0\| > 1 - \varepsilon$ for $\varepsilon > 0$. Let $S_1, \dots, S_p$ be slices of $\overline{C}_{n_0}$ such that $\mathrm{diam}(p^{-1}(S_1 + \cdots + S_p)) < \varepsilon$ and such that z_0 is in $p^{-1}(S_1 + \cdots + S_p)$. In particular, for every $z \in p^{-1}(S_1 + \cdots + S_p)$ we have $\|z\| > 1 - 2\varepsilon$.

For $1 \le i \le p$ define $T_i = \pi[1, n]^{-1}(S_i \cap C_{n_0})$ which are slices of C. For $x \in p^{-1}(T_1 + \cdots + T_p)$ we have $\|\pi[1, n](x)\| > 1 - 2\varepsilon$ hence $\|(\mathrm{Id} - \pi[1, n])(x)\| < 2 \cdot \varepsilon^{1/2}$. For $z_1, z_2 \in p^{-1}(T_1 + \cdots + T_p)$ we therefore obtain $\|z_1 - z_2\| < 4\varepsilon^{1/2} + \varepsilon$. The proposition is proved. □

Proposition II.4. *Strong regularity is stable under* G_δ*-embeddings.*

Proof. Let $T: X \mapsto Y$ be a G_δ-embedding into a strongly regular Banach space Y and C a closed, convex, bounded subset of X. By a classical result of Baire [BR2, Proposition 1.9] the points $y \in T(C)$ at which the restriction T^{-1} to $T(C)$ is continuous form a dense G_δ-subset of $T(C)$.

For $\varepsilon > 0$ we may therefore find $y \in T(C)$ and $\delta > 0$ such that $T^{-1}(T(C) \cap B(y, \delta))$ has diameter less than ε, where $B(y, \delta)$ denotes the ball of radius δ around y.

By the norm-density of S.C.S-points in $\overline{T(C)}$ we may find slices $S_1, \dots, S_n$ of $\overline{T(C)}$ such that $n^{-1}(S_1 + \cdots + S_n)$ is contained in $B(y, \delta)$. Letting $T_i = T^{-1}(S_i \cap T(C))$ for $1 \le i \le n$, we obtain slices $T_1, \dots, T_n$ of C such that the diameter of $n^{-1}(T_1 + \cdots + T_n)$ has diameter less than ε. The proposition is proved.

Remark II.5. Here are two other examples of properties stable under G_δ-embeddings and l_2-sums and hence stable under complemented boundedly complete skipped blocking decompositions (see [GR] for details).

(P_1) Every operator from L^1 into X is a *Dunford-Pettis operator*, i.e. its restriction to L^∞ is norm compact.

(P_2) No operator from L^1 into X is a *sign embedding* $T: L^1 \to X$ is a sign embedding if there exists $\delta > 0$ so that $\|Tf\| \geq \delta$ for each mean-zero function f verifying $|f| = 1$ a.s.

On the other hand, the Radon-Nikodým Property (equivalently every operator from L^1 into X is representable) is neither stable by G_δ-embedding [GM1] nor by boundedly complete skipped blocking decomposition [BR1]. However, we have the following.

Theorem II.6. *If a Banach space X admits a complemented l^1-skipped blocking decomposition into subspaces $(X_n)_n$ with the R.N.P, then X has the R.N.P.*

The proof is an adaptation of the arguments given in [BR1] for the case of finite dimensional components. We shall need the following standard result:

Lemma II.7. *Let $M_n: [0,1] \mapsto X$ be a bounded martingale adapted to an increasing sequence $(\Sigma_n)_{n=1}^\infty$ of σ-algebras generated by finite partitions of $[0,1]$ into sets of positive Lebesgue measure m and let C be the closed convex hull of $\{M_n(w): w \in [0,1], n \in \mathbf{N}\}$.*

For every slice S of C there is a set $A \subseteq [0,1]$, $m(A) > 0$ and $n_0 \in \mathbf{N}$ such that $\{M_n(w): w \in A, n \geq n_0\} \subseteq S$.

Proof. If S is of the form $S = S(x^*, \alpha) = \{x \in C: \langle x, x^*\rangle > N_{x^*} - \alpha\}$, where $x^* \in X^*$, $\|x^*\| = 1$, $\alpha > 0$ and $N_{x^*} = \sup\{\langle x, x^*\rangle: x \in C\}$. Find $n_0 \in \mathbf{N}$ and an atom B of Σ_{n_0} such that $\langle M_{n_0}(w), x^*\rangle > N_{x^*} - \alpha/2$ for $w \in B$.

Consider the real-valued martingale $(x^* \circ M_n)_{n \geq n_0}$ and let $\tau \geq n_0$ be the stopping time given by $\tau(w) = n$, if $x^* \circ M_{n_0}(w)$, $x^* \circ M_{n_0+1}(w), \dots, x^* \circ M_{n-1}(w)$ are greater than $N_{x^*} - \alpha$ and $x^* \circ M_n(w) \leq N_{x^*} - \alpha$ and $\tau(w) = \infty$ otherwise. Let $A = \{w \in B: \tau(w) = \infty\}$; for $w \in A$ we get from our construction that $M_n(w) \in S$ for all $n \geq n_0$ and we may estimate

$$\begin{aligned} m(B)(N_{x^*} - \alpha/2) < \int_B x^*(M_{n_0}(w))\, dm &= \int_B x^*(M_\tau(w))\, dm \\ &\leq m(A) \cdot N_{x^*} + (m(B) - m(A))(N_{x^*} - \alpha) \\ &= m(B)(N_{x^*} - \alpha) + m(A) \cdot \alpha, \end{aligned}$$

hence $m(A) > m(B)/2 > 0$. □

Proof of Theorem II.6. Let $\delta > 0$ be such that for every $r_1 < s_1 + 1 < r_2 < \cdots < r_k < s_k + 1$ and $x_i \in X[r_i, s_i]$ we have $\|\sum_{i=1}^k x_i\| < \delta \sum_{i=1}^k \|x_i\|$.

If X fails R.N.P then there is a bounded simple-valued martingale $M_n: [0,1] \mapsto X$ such that, for $w \in [0,1]$ and $n \in \mathbf{N}$, $\|M_{n+1}(w) - M_n(w)\| \geq 1$ (see [DU, p. 132]). Let C be the closed convex hull of $\{M_n(w): w \in [0,1], n \in \mathbf{N}\}$.

By Theorem II.1 above, X is strongly regular, hence there are slices $S_1, \dots, S_k$ of C such that $\operatorname{diam}(k^{-1}(S_1 + \cdots + S_k)) < \delta/4$.

By the preceding lemma we may find sets $A_1, \dots, A_k$ in $[0,1]$, $m(A_i) > 0$, and $n_0 \in \mathbf{N}$ such that, for $n \geq n_0$ and $1 \leq i \leq k$, $M_n(w) \in S_i$ for $w \in A_i$.

For $r \leq s$ denote by $P_{[r,s]}$ the projection onto $X[r,s]$. Using the hypothesis that, for every $s \in \mathbf{N}$, the range of $P_{[1,s]}$ has R.N.P it is routine to construct by induction on $1 \leq i \leq k$, points $w_i \in A_i$ and integers n_i, r_i, s_i such that $n_i \geq n_0$ and $1 = r_1 < s_{1+1} < r_2 < \cdots < r_k < s_{k+1}$ such that

$$\|(\mathrm{Id} - P_{[r_i,s_i]})(M_{n_i+1}(w_i) - M_{n_i}(w_i))\| < \delta/4.$$

We may estimate

$$\begin{aligned}\left\| k^{-1}\sum_{i=1}^{k} M_{n_i+1}(w_i) - k^{-1}\sum_{i=1}^{k} M_{n_i}(w_i)\right\| \\ \leq \left\| k^{-1}\sum_{i=1}^{k} P_{[r_i,s_i]}(M_{n_i+1}(w_i) - M_{n_i}(w_i))\right\| - \delta/4 \\ \leq \delta(1-\delta/4) - \delta/4 > \delta/4.\end{aligned}$$

On the other hand, $k^{-1}\sum_{i=1}^{k} M_{n_i+1}(w_i)$ and $k^{-1}\sum_{i=1}^{k} M_{n_i}(w_i)$ are elements of $k^{-1}(S_1 + \cdots + S_k)$, hence

$$\left\| k^{-1}\sum_{i=1}^{k} M_{n_i+1}(w_i) - k^{-1}\sum_{i=1}^{k} M_{n_i}(w_i)\right\| < \delta/4,$$

a contradiction, finishing the proof. □

Remark II.8. By similar (even slightly simpler) arguments one obtains an analogous result to II.6 for P.C.P and C.P.C.P: If X admits an l^1-skipped decomposition into spaces $(X_n)_n$ having P.C.P (resp. C.P.C.P) then X has P.C.P (resp. C.P.C.P). We shall give a better result in §IV.

III. Uniformizing maps and the uniform C.P.C.P.

If a Banach space X has the C.P.C.P, it is easy to see that for every closed convex bounded subset D of X the set pc(D) consisting of the points of continuity for the identity map $\mathrm{Id}\colon (D, \text{weak}) \to (D, \|\ \|)$ is *weakly dense* in D. In general, and unlike the set of S.C.S points [GGMS], pc(D) fails to be norm dense in D. Indeed, if D is the unit ball of l_2 then pc(D) is nothing else but the unit sphere. This is the reason why the P.C.P [GMS1] and the C.P.C.P (§VI) are not stable under *G_δ-embeddings* nor under *boundedly complete skipped blocking sums* while the property of strong regularity is stable under such operations (see the proof of Theorem II.4).

To remedy this situation, we shall consider a case where pc(D) is dense in D for a norm $|||\ |||$ that is weaker than the original one $\|\ \|$ but not much weaker since we require that $\{x \in X\colon \|x\| \leq 1\}$ be $|||\ |||$-closed. Here is the needed concept.

Definition III.1. Let C be a closed convex bounded subset of a Banach space X. We shall say that C has the *uniform convex point of continuity property* (uniform C.P.C.P) if there is a semiembedding $S\colon X \to c_0$ such that for any closed convex bounded subset D of C the set $S(\mathrm{pc}(D))$ is norm dense in $S(D)$.

We shall say that the Banach space X has the uniform C.P.C.P if its unit ball does. We may sometimes emphasize the operator involved, S, and say that (X, S) has uniform C.P.C.P.

Remark III.2. In the case of the unit ball D of l^2 mentioned above, note that the unit sphere (or $\mathrm{pc}(D)$) is actually dense in D for the c_0-norm. Actually, this holds for any closed convex bounded subset of l^2. In other words (l^2, S) has uniform C.P.C.P with S being the canonical injection into c_0. This will follow from Theorem III.7 below.

Definition III.3. (i) An operator $S\colon X \to Y$ is said to be *uniformizing* if for every convex bounded subset $D \subseteq X$, $\varepsilon > 0$, and y in $S(D)$ there exists a weak open set W in X with $W \cap D \neq \varnothing$ such that $S(W \cap D)$ is contained in the ball $B(y, \varepsilon)$ centered at y with radius ε.

(ii) It is said to be *convex huskable* if for every subset $D \subset X$, and any $\varepsilon > 0$, there exists a weak open set W with $W \cap D \neq \varnothing$ and $\mathrm{diam}(S(W \cap D)) \leq \varepsilon$.

Remark III.4. Denote by $\mathrm{pc}(S, D)$ the set of points of continuity for $S\colon (D, w) \to (S(D), \|\ \|)$. It is clear that S is convex huskable (resp. uniformizing) whenever $S(\mathrm{pc}(S, D))$ is nonempty (resp. is norm dense in $S(D)$) for any closed convex bounded D in X. It follows that any operator that is weak to norm continuous on bounded sets (in particular any compact operator) is uniformizing.

The proof of the following easy proposition is left to the interested reader.

Proposition III.5. *Let S be a one-to-one operator from X into Y. The following assertions are equivalent*:

(a) *X has the C.P.C.P and S is uniformizing.*
(b) *For any closed convex bounded subset D of X the set $S(\mathrm{pc}(D))$ is norm dense in $S(D)$.*
(c) *For any closed convex bounded subset D of X, $\varepsilon > 0$ and y in $S(D)$, there exists a weak open set W in X with $W \cap D \neq \varnothing$, $\mathrm{diam}(W \cap D) \leq \varepsilon$ and $S(W \cap D) \subset B(y, \varepsilon)$.*

Remark III.6. It follows that a Banach space X has *uniform* C.P.C.P if and only if X has C.P.C.P and there exists a uniformizing semiembedding of X into c_0. We shall now show that the "uniformizing condition" can be omitted since it will be automatically satisfied. The following theorem is the main result of this section.

Theorem III.7. *Let X be a Banach space with the C.P.C.P and let K be a countable compact space. Then every one-to-one bounded operator from X into $C(K)$ is uniformizing.*

The proof of this result will follow from a more general principle that we emphasize since it might be of independent interest. The theorem itself follows from III.14.

Let X be a Banach space. For any bounded subset A of X^*, denote by τ_A the topology on X of uniform convergence on A. A typical τ_A-neighborhood of zero will then be a set of the form

$$V(A, \varepsilon) = \left\{ x \in X ;\ \sup_{x^* \in A} |x^*(x)| < \varepsilon \right\}, \quad \text{where } \varepsilon > 0 .$$

Definition III.8. Let D be a convex bounded subset of a Banach space X and let A be a bounded subset of X^*. We shall say that

(i) A is a *uniformizing set* for D if for every convex subset $F \subseteq D$, x in F and $\varepsilon > 0$, there exists a weak open set W with $W \cap F \neq \emptyset$ and $W \cap F \subset x + V(A, \varepsilon)$.

(ii) D is $(w - \tau_A)$ *convex huskable* if for every convex subset $F \subseteq D$ and $\varepsilon > 0$, there exists a weak open set W with $W \cap F \neq \emptyset$ and $W \cap F - W \cap F \subset V(A, \varepsilon)$. Here "$-$" denotes the algebraic set difference.

Remark III.9. It is clear that (ii) is in general weaker than (i). Indeed, if τ_A is generated by the norm ($A = \text{Ball}(X^*)$), (ii) is then equivalent to D having the C.P.C.P, while (i) requires that the *points of weak to norm continuity* pc(D) be norm dense in D which does not hold even in l^2.

We shall summarize, in the following lemma, the various properties of uniformizing sets that are needed in the proof of the next proposition.

Lemma III.10. *Let D be a convex bounded subset of a Banach space X and let A be a bounded subset of X^* then:*

(a) *If A is uniformizing for D, then $\overline{A}^*$ is also uniformizing for D.*

(b) *Any finite set L in X^* is uniformizing for D. Furthermore, $A \cup L$ is also uniformizing whenever A is.*

(c) *If B is a subset of A that is uniformizing for D and if D is $(w - \tau_A)$ convex huskable, then for any convex subset $F \subset D$, x in F and $\varepsilon > 0$, there exists a weak open set W with $W \cap F \neq \emptyset$, $W \cap F \subset x + V(B, \varepsilon)$ and $W \cap F - W \cap F \subset V(A, \varepsilon)$.*

The proof of Lemma III.10 is straightforward and left to the reader.

We introduce one more definition: we say that $B \subset X^*$ is *strongly uniformizing* for D if $B \cup K$ is uniformizing for D whenever $K \subset X^*$ is uniformizing for D. Note that finite sets are strongly uniformizing by III.10(b). We shall now prove the following proposition.

Proposition III.11. *Let D be a convex bounded subset of a Banach space X and let B be a w^*-compact subset of X^* such that D is $(w - \tau_B)$ convex huskable. Suppose that there exists a w^*-closed subset B_0 of B such that B_0 is strongly uniformizing for D and such that every w^*-closed subset L of B*

which is disjoint from B_0 *is also strongly uniformizing for* D. *Then* B *is strongly uniformizing for* D.

Proof. Let K be a w^*-compact subset of X^*, uniformizing for D. Consider $A = B \cup K$. It follows immediately from the hypothesis that D is $(w - \tau_A)$-convex huskable. Let $A_0 = B_0 \cup K$. Then A_0 is uniformizing for D; if L is a w^*-closed subset of A such that $L \cap A_0 = \varnothing$, then L is contained in B and is disjoint from B_0, thus it is strongly uniformizing for D by hypothesis. It remains to show that A is uniformizing for D.

We may suppose without loss of generality that $\sup\{|a(x-y)| : a \in A, x, y \in D\} \leq 1$. Let C be a convex subset of D, let $x \in C$, $\theta \in]0,1]$ and set $C_\theta = \{y \in C : y - x \in V(A, \theta)\}$. We need to show that for any $\theta \in]0,1]$, there exists a weak open set W with $W \cap C \neq \varnothing$ and $C \cap W \subset C_\theta$.

Suppose not and let $\delta \in]0, \frac{1}{2}]$ such that:

(∗) For every weak open set W, $C \cap W \subset C_{2\delta}$ implies that $C \cap W = \varnothing$.

We shall work toward a contradiction. Note first that Lemma III.10(c) applied to $A_0 \subset A$ and C_δ gives a weakly open convex set W such that

$$(**)\qquad \begin{gathered} C_\delta \cap W \neq \varnothing, \quad C_\delta \cap W \subset x + V(A_0, \delta/8) \quad \text{and} \\ C_\delta \cap W - C_\delta \cap W \subset V(A, \delta^2/8). \end{gathered}$$

Let z be an element in $C_\delta \cap W$. By replacing z if necessary with an element in the segment $]x, z[$, we can assume that $\rho = \sup\{|a(z-x)| : a \in A\} < \delta$. Note that $L_0 = \{a \in A : |a(z-x)| \geq \delta/4\}$ is a w^*-closed subset of A that is disjoint from A_0 by (∗∗). It follows from the above discussion that L_0 is strongly uniformizing for D. Now let $\varepsilon > 0$ be such that:

$$(***)\qquad 2\varepsilon \leq \delta^2/8 \quad \text{and} \quad \rho + 2\varepsilon \leq \delta.$$

Choose an integer N so that $\delta/4 \leq 1/N \leq \delta/2$. We can construct inductively vectors $z_1, z_2, \dots, z_N$ in $F = C \cap W$, convex weak open sets $W_1, \dots, W_N$ and pairwise disjoint w^*-compact subsets $L_1, \dots, L_N$ of $A \backslash A_0$ such that for $1 \leq j \leq N$:

(i) $W_j \cap F \neq \varnothing$ and $W_j \cap F \subset z + V(A_j, \varepsilon)$ where $A_j = A_0 \cup L_0 \cup \cdots \cup L_{j-1}$,
(ii) $z_j \in (F \cap W_j) \backslash C_{2\delta}$,
(iii) $L_j = \{a \in A : |a(z_j - z)| \geq 2\varepsilon\}$ is strongly uniformizing for D.

Start with L_0. We have seen that L_0 is strongly uniformizing for D, thus $A_0 \cup L_0$ is uniformizing for D, hence there exists a convex weak open set W_1 such that (i) is verified with $j = 1$. (∗) applied to $W \cap W_1$ gives (ii). The set $L_1 = \{a \in A : |a(z_1 - z)| \geq 2\varepsilon\}$ is w^*-compact and disjoint from A_0 because of (i), hence L_1 is strongly uniformizing for D. The general step of the induction is similar and is left to the reader.

Note that by (iii) we have for each $1 \leq j \leq N$:

$$(1)\qquad \sup\{|a(z_j - z)| : a \in A \backslash L_j\} \leq 2\varepsilon.$$

Hence by the definition of ρ,

$$\sup\{|a(z_j - x)| : a \in A \setminus L_j\} \leq 2\varepsilon + \rho . \tag{2}$$

Since $z_j \notin C_{2\delta}$ we have

$$\sup\{|a(z_j - x)| : a \in A\} > 2\delta . \tag{3}$$

By $(***)$ we have $2\varepsilon + \rho \leq \delta$, hence (2) and (3) give

$$\sup\{|a(z_j - x)| : a \in A\} = \max\{|a(z_j - x)| : a \in L_j\} . \tag{4}$$

Moreover, since $L_j \cap L_0 = \varnothing$ we have for $1 \leq j \leq N$

$$\max\{|a(z - x)| : a \in L_j\} \leq \delta/4 . \tag{5}$$

Combining (5), (4) and (3) we obtain

$$\max\{|a(z_j - z)| : a \in L_j\} > 7\delta/4 . \tag{6}$$

If we let $z' = N^{-1}(z_1 + \cdots + z_N)$ we get

$$\max\{|a(z' - z)| : a \in A\} \geq 7\delta/4N - 2\varepsilon(N-1)/N > 7\delta^2/16 - \delta^2/8 .$$

Hence

$$\max\{|a(z' - z)| : a \in A\} \geq \delta^2/4 . \tag{7}$$

To get the required contradiction, it is enough to prove that $z' \in C_\delta \cap W$ because then by $(**)$ $\max\{|a(z' - z)| : a \in A\} \leq \delta^2/8$ which contradicts (7).

Note first that by convexity $z' \in C \cap W$. Moreover,

$$\sup\left\{|a(z' - z)| : a \in A \setminus \bigcup_{j=1}^{N} L_j\right\} \leq 2\varepsilon . \tag{8}$$

Hence

$$\sup\left\{|a(z' - x)| : a \in A \setminus \bigcup_{j=1}^{N} L_j\right\} \leq 2\varepsilon + \rho \leq \delta . \tag{9}$$

On the other hand for each $1 \leq j \leq N$

$$\sup\{|a(z_j - z) : a \in L_j\} \leq 1 . \tag{10}$$

It follows that

$$\sup\{|a(z' - z)| : a \in L_j\} \leq \frac{1}{N} + \frac{N-1}{N} \cdot 2\varepsilon \leq \delta/2 + \delta/8 . \tag{11}$$

But since $L_0 \cap L_j = \varnothing$

$$\sup\{|a(z - x)| : a \in L_j\} \leq \delta/4 . \tag{12}$$

(11) and (12) give

$$\sup\{|a(z' - x)| : a \in L_j\} \leq \delta . \tag{13}$$

Finally (13) combined with (9) gives that $z' \in C_\delta$. This finishes the proof of the proposition.

Remark III.12. The above proof does not require the "uniformizing" and "huskability" conditions on all convex subsets of D. Only the convex relatively weak-open set in D are involved. We then obtain that B is uniformizing on the convex relatively weak open sets in D.

Let A be a w^*-compact subset of X^*. As is classical we denote by A' the derived set of A equipped with the w^*-topology, that is the set of w^*-accumulation points of A; let $A^{(0)} = A$ and define for every ordinal $\alpha > 0$, $A^{(\alpha)} = (A^{(\beta)})'$ if $\alpha = \beta + 1$ and $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ if α is a limit ordinal.

Corollary III.13. *Let X be a Banach space, K a w^*-compact subset of X^* and D a convex bounded subset of X that is $(w - \tau_K)$-convex huskable. If $K^{(\alpha)}$ is strongly uniformizing for D for some ordinal α, then K is strongly uniformizing for D.*

Proof. We start by proving a weaker statement: if $K^{(\alpha)}$ is finite then K is strongly uniformizing for D. The proof works by transfinite induction on the smallest α such that $K^{(\alpha)}$ is finite. When $\alpha = 0$ the set K itself is finite and this case is given by III.10(b). Suppose now that the property is true for every $\beta < \alpha$ and assume that $K^{(\alpha)}$ is finite. Then apply Proposition III.11 to $B = K$ and $B_0 = K^{(\alpha)}$. We have that B_0 is strongly uniformizing for D, and if L is a w^*-closed subset of B such that $L \cap B_0 = \varnothing$ it follows that $L^{(\alpha)} = \varnothing$, therefore L is strongly uniformizing for D by the induction hypothesis; hence K is strongly uniformizing for D by III.11, and our first step is done.

Assume that $K^{(\alpha)}$ is strongly uniformizing for D. Apply again Proposition III.11 with $B = K$ and $B_0 = K^{(\alpha)}$. As before, if L is a w^*-closed subset of K such that $L \cap B_0 = \varnothing$, we have $L^{(\alpha)} = \varnothing$ thus L is strongly uniformizing by the first step and a last application of III.11 shows that K is strongly uniformizing for D.

Corollary III.14. *Let T be a bounded linear operator from a Banach space X into $C(K)$ where K is countable. If T is convex huskable, then it is a uniformizing operator.*

Proof. Apply Corollary III.13 to the set $K_0 = \{T^*\delta_k : k \in K\}$ in X^* and note that since it is countable we have that $K_0^{(\alpha)}$ is empty for a large enough α.

Remark III.15. (a) It is clear that Corollary III.14 implies Theorem III.7 since $\mathrm{pc}(D) \subset \mathrm{pc}(T, D)$ for any set D in X and any bounded linear operator.

Corollary III.16. (a) *A Banach space X has uniform C.P.C.P if and only if it has C.P.C.P and there exists a semiembedding from X into c_0.*

(b) *A Banach space with P.C.P has uniform C.P.C.P.*

Proof. (a) is immediate from Theorem III.7 while (b) follows from (a) and Theorem IV.1 of [GM2].

IV. JT_∞-TYPE DECOMPOSITIONS AND THE ITERATED JAMES-TREE SPACES

We shall start by defining the concept of JT_∞-type decomposition. It is done in such a way that it coincides with the notion of JT-type decomposition whenever the factor spaces are finite dimensional.

Definition IV.1. A sequence $(X_n)_{n=1}^\infty$ of closed subspaces of a Banach space X is called a *JT_∞-type decomposition* for X if:

(i) $(X_n)_{n=1}^\infty$ is a complemented boundedly complete skipped blocking decomposition of X with associated projections $p_n: X \to X_n$ verifying $p_n p_m = 0$ if $m \neq n$.

(ii) There is an injection $S: X \to c_0(I)$ and a partition of I into countable subsets $(I_n)_{n=1}^\infty$ such that

(α) $\pi_n \circ S = S \circ p_n = S_n$ for each n, where $\pi_n: c_0(I) \to c_0(I_n)$ denotes the natural coordinate projections, and

(β) for any x^{**} in X^{**} verifying $\pi_n^{**} \circ S^{**} x^{**} \in c_0$ for each $n \in \mathbf{N}$ and $\liminf_{n\to\infty} \|\pi_n^{**} \circ S^{**} x^{**}\| = 0$, there exists $x \in X$, $\|x\| \leq \|x^{**}\|$ and $Sx = S^{**}x^{**}$.

S will be called the *JT_∞-injection associated* to $(X_n)_n$. We shall say that $(X_n)_n$ is a *compact* (resp. *uniformizing*) *JT_∞-type decomposition for* X if in addition the restriction of S to any finite sum of the X_n's is compact (resp. uniformizing).

For obvious reasons, we shall sometimes say that $(X_n, S_n)_n$ is a JT_∞-type decomposition for (X, S). Note that S (as well as each S_n) is automatically a semiembedding of X (of X_n) into c_0.

Remark IV.2. The JT_∞- and the JT-type decompositions coincide when the factor spaces $(X_n)_n$ are finite dimensional. There is an obvious correspondence between operators $S_n: X_n \to l_{m_n}^\infty$ and the families of vectors $(y_n^i)_{i=1}^{m_n}$ involved in Definition I.1 of a JT-type decomposition. The operator S becomes a JT-injection (Theorem I.7). In (β), the condition $\pi_n^{**} \circ S^{**} x^{**} \in c_0$ becomes irrelevant whenever X_n is reflexive, or—a fortiori—finite dimensional.

As in the case of a JT-type decomposition (Proposition I.3), the situation becomes easier when we have a Schauder decomposition.

Proposition IV.3. *Let $(X_n)_n$ be a Schauder decomposition for a separable Banach space X. The following conditions are then equivalent:*

(a) *$(X_n)_n$ is a JT_∞-type decomposition for X.*

(b) *There exists a semiembedding $S: X \to c_0(I)$ with $\pi_n \circ S = S \circ p_n$ for all $n \in \mathbf{N}$, such that the series $\Sigma_{n=1}^\infty x_n$ converges in X whenever $x_n \in X_n$, $\sup_n \|\Sigma_{i=1}^n x_i\| < \infty$ and $\liminf_n \|S(x_n)\| = 0$.*

Proof. (a) $\Rightarrow$ (b) Let $x_n \in X_n$ and let $x^{**} \in X^{**}$ be a w^*-cluster point of the partial sums $s_n = \sum_{i=1}^n x_i$. Assumption (a) gives an x in X with $\|x\| \leq \|x^{**}\|$ such that $p_n(x) = x_n$. Since $(X_n)_n$ is a Schauder decomposition we get that $(S_n)_n$ norm converges to x.

(b) $\Rightarrow$ (a) Let $x^{**} \in X^{**}$, $\|x^{**}\| \le 1$, such that $\pi_n^{**} \circ S^{**}x^{**} \in c_0$. For $n \in \mathbf{N}$, $z_n = \sum_{i=1}^n \pi_n^{**} \circ S^{**}x^{**}$ belongs to the norm closure of $S \circ Q_n(\mathrm{ball}(X))$ where $Q_n = \sum_{i=1}^n p_i$. Hence there is a unique $s_n \in Q_n(X)$ such that $S(s_n) = z_n$ and $\|s_n\| \le C$ where C is the basis constant of the Schauder decomposition. On the other hand $x_n = s_n - s_{n-1}$ verifies the assumptions of (b), hence $(s_n)_n$ converges to some x in X with $\|x\| \le C$. But since $S(x) = S^{**}(x^{**})$ belongs to the closure of $S(\mathrm{ball}(X))$, we deduce that $\|x\| \le 1$ since S is a semiembedding.

Remark IV.4. Note that (b) implies in particular that $(X_n)_{n=1}^\infty$ is a complemented boundedly complete skipped blocking decomposition. However, unlike the case where the X_n's are finite dimensional, they can form a complemented boundedly complete skipped blocking decomposition which is not of the JT_∞-type, that is, without inducing the appropriate semiembedding into c_0. (See the space $S_* T_\infty$ constructed in §VI.)

The following theorem is the main result of this section.

Theorem IV.5. *Let $(X_n)_n$ be a JT_∞-type decomposition for a separable Banach space X.*

(1) *If each X_n has P.C.P and the decomposition is compact then X has P.C.P.*
(2) *If each X_n has C.P.C.P then X has C.P.C.P.*

In both cases X has the uniform C.P.C.P.

We shall deduce the theorem from the following more precise result, combined with Theorem III.7

Proposition IV.6. *Let $(X_n)_n$ be a complemented skipped blocking decomposition of a separable Banach space X with associated projections $p_n\colon X \to X_n$ verifying $p_n p_m = 0$ if $m \ne n$. Let $S\colon X \to c_0(\bigcup_{n=1}^\infty I_n)$ be an injection such that $S \circ p_n = \pi_n \circ S = S_n$, where the π_n are the coordinate projections on the disjoint sets I_n. Suppose that for any x^{**} in X^{**} verifying*

(i) $\pi_n^{**} \circ S^{**}x^{**} = S_n x_n$ *for some x_n in X_n $(n = 1, 2, \dots)$ and*
(ii) $\liminf_{n\to\infty} \|S_n x_n\| = 0$

there exists x in X with $\|x\| \le \|x^{**}\|$ and $p_n(x) = x_n$ for each n. Then:

(a) *If S_n is a semiembedding on X_n for each n, then S is a semiembedding on X.*
(b) *If also each S_n is uniformizing on X_n for all n, then S is uniformizing on X.*
(c) *If in addition each X_n has C.P.C.P for all n, then X has C.P.C.P as well.*
(d) *If each X_n has P.C.P and S_n is compact for all n, then X has P.C.P.*

It follows that if each (X_n, S_n) has uniform C.P.C.P then (X, S) has uniform C.P.C.P.

Remark IV.7. (1) Hypothesis (i) in Proposition IV.6 is stronger than the corresponding one in (β) of Definition IV.1 (for JT_∞-type decomposition). However, they become equivalent when each S_n is assumed to be a semiembedding.

(2) The hypothesis (i) in Definition IV.1 is not needed for the stability of C.P.C.P under JT_∞-type decompositions. We do not know if it is already implied by (ii). Note that it is the case if the X_n 's form a Schauder decomposition.

(3) Note that even if each S_n is compact and each X_n has P.C.P, we only obtain that S is uniformizing and not necessarily compact. To see that, it is enough to take any JT-injection into c_0 associated—by Theorem I.10—to a space with the P.C.P but which is not a separable dual.

Proof of Proposition IV.6. (a) is immediate. To prove (b), assume that W is an open ball of c_0 such that $S(C) \cap W \neq \varnothing$, where C is a norm closed convex bounded subset of X. We shall show the existence of a weakly open set V of X with $V \cap C \neq \varnothing$ and $S(V \cap C) \subseteq W$ (i.e. S is uniformizing).

Note first that, modulo a translation and a multiplication by a scalar, we may assume without loss of generality that C contains the origin and that W equals the open unit ball of c_0.

Assuming our claim false we shall work towards a contradiction: By assumption there is $x_1 \in C$ such that $\|Sx_1\| \geq 1$. By the convexity of C and the fact that C contains the origin we may find $m_1 \in \mathbf{N}$ and $0 < \lambda < 1$ such that, for $z_1 = \lambda x_1$ in C, we have $\|Sz_1\| < 1$ and $\|\pi_{m_1} S z_1\| > 1/2$. Find $n_1 > m_1$, such that $\|\pi_n S z_1\| < 2^{-1}$ for $n \geq n_1$ and define the open convex set

$$W_1 = \{y \in c_0 \colon \|\pi_n(y - Sz_1)\| < 2^{-1} \text{ and } \|\pi_n(y)\| < 1 \text{ for } n \leq n_1\}$$

which clearly contains Sz_1.

Assume that we have constructed elements $z_1, \dots, z_k$ in C, integers $m_1 < n_1 < m_2 < \cdots < n_k$ and open convex subsets $W_1, \dots, W_k$ of c_0 such that, for $1 \leq i \leq k$,

(i) $\|Sz_i\| < 1$,
(ii) $\|\pi_{m_i} S z_i\| > 1/2$,
(iii) $\|\pi_n S z_i\| < 2^{-i}$ for $n \geq n_i$,
(iv) $W_i = W_{i-1} \cap \{y \in c_0 \colon \|\pi_n(y - Sz_i)\| < 2^{-i} \text{ and } \|\pi_n(y)\| < 1 \text{ for } n \leq n_i\}$,
(v) $Sz_i \in W_i$.

For the inductive step note first that $\sum_{i=1}^{n_k} S_i$ is convex-huskable and hence uniformizing on $\sum_{i=1}^{n_k} \bigoplus X_i$ by Corollary III.14. Hence we can choose a weakly open convex set U_{k+1}, $U_{k+1} \cap C = \varnothing$, such that

$$(*) \qquad S(U_{k+1} \cap C) \subseteq W_k .$$

This is possible as W_k depends only on $(I_n)_{n \leq n_k}$.

By assumption there is $x_{k+1} \in U_{k+1} \cap C$ such that $\|Sx_{k+1}\| \geq 1$. Noting (i) and since by (iv) and ($*$) we have $\|\pi_n x_{k+1}\| < 1$ for $n \leq n_1$, we may find a suitable $0 < \lambda < 1$ and $m_{k+1} > n_k$ such that for $z_{k+1} = \lambda x_{k+1} + (1 - \lambda) z_k$,

(i) and (ii) hold true for $i = k+1$. Note that $S(x_{k+1})$ and $S(z_k)$ are in W_k, hence by convexity $Sz_{k+1} \in W_k$. Finally find $n_{k+1} > m_{k+1}$ such that (iii) holds true for $i = k+1$ and define W_{k+1} by (iv). Clearly (v) holds for $i = k+1$. Note that $W_{k+1} \cap S(C)$ is nonempty as $S(z_{k+1}) \in W_{k+1}$. This completes the inductive step.

Let x^{**} be a w^*-cluster point of $(z_k)_{k=1}^\infty$ in X^{**}. Note that it follows from (iv) and (v) that for every $n \in \mathbf{N}$ and k big enough $\|\pi_n \circ S(z_{k+1} - z_k)\| < 2^{-k}$, hence $\pi_n \circ S(z_k)$ converges in c_0 to $\pi_n^{**} \circ S^{**}(x^{**})$. Since $S_n = \pi_n \circ S$ is a semiembedding, there exists x_n in X_n with $\pi_n^{**} \circ S^{**}(x^{**}) = S_n x_n$ for $n \in \mathbf{N}$.

Similarly it follows from (ii), (iv), and (v) that

$$\liminf_{i \mapsto \infty} \|\pi_{n_i}^{**} S^{**}(x^{**})\| = 0.$$

Hence from the hypothesis there is $x \in X$ such that $S(x) = S^{**}(x^{**})$, whence in particular $S^{**}(x^{**}) \in c_0$. This is contradictory to the fact that

$$\liminf_{i \mapsto \infty} \|\pi_{m_i}^{**} S^{**}(x^{**})\| \geq 1/2,$$

which follows from (ii), (iv) and (v). This shows that S is uniformizing.

(c) To prove that X has C.P.C.P, let C be a closed convex bounded subset of X. Assuming there is an $\varepsilon > 0$ such that for every weakly open set U with $U \cap C \neq \varnothing$ we have $\operatorname{diam}(U \cap C) > \varepsilon$, let us work towards a contradiction.

Denote $Q_n = \sum_{i=1}^n p_i$ and let $(x_k)_{k=1}^\infty$ be a dense sequence in X. Denote by B_k the closed ball of radius $\varepsilon/2$ and centered at x_k. We construct inductively an increasing sequence of integers $(n_k)_{k=1}^\infty$ and a decreasing sequence $(U_k)_{k=1}^\infty$ of convex weakly open sets such that, for every k,

(i) $U_k \cap C \neq \varnothing$,
(ii) $\overline{S(U_k)} \cap S(B_k) = \varnothing$,
(iii) $\sup\{\|Sp_{n_k}(x)\| : x \in U_k \cap C\} \leq k^{-1}$,
(iv) $\operatorname{diam}(Q_{n_k}(C) \cap U_k) < 2^{-k}[\sup\{\|p_j\| : j \leq n_k]^{-1}$.

We only give the inductive step: If $(n_i)_{i \leq k}$ and $(U_i)_{i \leq k}$ are defined up to k, note that $U_k \cap C$ has diameter larger than ε and therefore cannot be contained in B_{k+1}. Choose $x_{k+1} \in (U_k \cap C) \setminus B_{k+1}$. As S is a semiembedding, $S(B_{k+1})$ is closed and $Sx_{k+1} \notin S(B_{k+1})$, we may find an open half-space H_{k+1} of c_0 containing $S(x_{k+1})$ and such that the closure of H_{k+1} is disjoint from $S(B_{k+1})$.

Let $C_k = U_k \cap C \cap S^{-1}(H_{k+1})$ and find $n_{k+1} > n_k$ such that

(α) $\|Sp_{n_{k+1}}(x_{k+1})\| < 1/2k$.

Since the restriction to S of $Q_{n_{k+1}}(X)$ is uniformizing and since the latter has C.P.C.P, we can find a convex weak open set V_{k+1} in $Q_{n_{k+1}}(X)$ such that:

(β) $V_{k+1} \cap Q_{n_{k+1}}(C_k) \neq \varnothing$.
(γ) $\operatorname{diam}(V_{k+1} \cap Q_{n_{k+1}}(C_k)) \leq 2^{-k}[\sup\{\|p_j\| : j \leq n_k]^{-1}$, and
(δ) $SQ_{n_{k+1}}(V_{k+1} \cap Q_{n_{k+1}}(C_k)) \subset B(SQ_{n_{k+1}}(x_{k+1}), 1/2k)$.

Let $U_{k+1} = Q_{n_{k+1}}^{-1}(V_{k+1}) \cap S^{-1}(H_{k+1}) \cap U_k$. Note that (β) implies that (i) is verified: $U_{k+1} \cap C \neq \emptyset$. (ii) is clear since $U_{k+1} \subseteq S^{-1}(H_{k+1})$. (iii) follows from (α) and (δ). While (iv) is a consequence of (γ). This finishes the inductive construction.

Now let $x^{**} \in X^{**}$ be in the nonempty intersection of the weak*-closures of $(U_n \cap C)_{n=1}^{\infty}$. We get from (iv) that $p_n^{**}x^{**} = x_n \in X_n$ and from (iii) that $\liminf_n \|\pi_n^{**}S^{**}(x^{**})\| = 0$.

By the hypothesis, there exists x in X with $S(x) = S^{**}(x^{**}) \in \overline{S(U_k)}^* \cap c_0 = \overline{S(U_k)}$ for each k. On the other hand, there exists necessarily a k_0 so that $S(x) \in S(B_{k_0})$. This clearly contradicts (ii) and finishes the proof of (c).

The proof of (d) is similar to (c). It is enough to remove the convexity assumption on C and to use the facts that $X_1 \oplus X_2 \oplus \cdots \oplus X_n$ has P.C.P. and SQ_n is compact to get that for any closed bounded subset C of X, any $\varepsilon > 0$ and any x in C, there is a weak neighborhood V_n with $V_n \cap Q_n(C) \neq \emptyset$, $\operatorname{diam}(V_n \cap Q_n(C)) \leq \varepsilon$ and $SQ_n(V_n \cap Q_n(C)) \subseteq B(SQ_n(x), \varepsilon)$. □

Actually the full strength of the compactness assumption on the factors S_n is not needed. It is enough that S_n verifies the "uniformization condition" on all and not only the convex bounded sets of X_n. □

We now apply the results of this section to show that the spaces $J_*T_{\infty,n}$ constructed in [GM2] have uniform C.P.C.P for all $n \in \mathbf{N}$ (in fact this holds true for all countable ordinals). The space $J_*T_{\infty,1}$ (or B_∞) was shown to have the C.P.C.P in [GMS]. For the convenience of the reader, we shall summarize below the properties of these spaces that are relevant to our study.

Theorem IV.8. *For each integer $n = 0, 1, \ldots$, there exists a Banach space—denoted $J_*T_{\infty,n}$—with the following properties:*

(a) $J_*T_{\infty,0} = l^2$.
(b) *The dual-denoted $JT_{\infty,n}$ is separable for each n.*
(c) *The quotient $JT_{\infty,n}^*/J_*T_{\infty,n}$ is isometric to $l_2(\Gamma)$ where Γ is uncountable whenever $n \geq 1$.*
(d) *For each $n \geq 0$, $J_*T_{\infty,n+1}$ has a JT_∞-type decomposition into subspaces isometric to $l_2(J_*T_{\infty,n})$.*
(e) *For each $n \geq 0$, $J_*T_{\infty,n}$ has the uniform C.P.C.P.*
(f) *For each $n \geq 0$, $J_*T_{\infty,n+1}$ can be mapped into l_2 via the composition of $(n+1)$ but not n G_δ-embeddings.*
(g) *$J_*T_{\infty,n}$ fails the P.C.P for each $n \geq 1$.*
(h) *For $n \geq 2$, $J_*T_{\infty,n}$ has no complemented boundedly complete skipped blocking decomposition into reflexive subspaces.*

We shall sketch here the construction of these spaces and we refer the reader to [GM2] for more details.

Let $T_\infty = \bigcup_{k=1}^{\infty} \mathbf{N}^k$ and consider $\mathscr{X} = (X_t, D_t)_{t \in T_\infty}$ a family of couples where each X_t is a Banach space and D_t a countable subset of the unit sphere of X_t. We will consider the elements of D_t as indexed by the immediate successors of t in T_∞: i.e. $D_t = \{d_{t,i} : i \in \mathbf{N}\}$.

Denote by $l^2(\mathscr{X})$ and $l^2(\mathscr{X}^*)$ respectively the spaces $(\sum_{t\in T_\infty} \oplus X_t)_{l_2}$ and $(\sum_{t\in T_\infty} \oplus X_t^*)_{l_2}$, and let j_t be the canonical injection of X_t into $l^2(\mathscr{X})$. If $S = \{t_0, t_1, \dots, t_n\}$ is a segment in T_∞ and $y \in \text{ball}(X_{t_n})$ define the atom $a_{S,y}$ to be

$$a_{S,y} = \sum_{k=0}^{n-1} j_{t_k}(d_{t_{k+1}}) + j_{t_n}(y),$$

then we say that S is the support of this atom.

Define for x^* in $l^2(\mathscr{X}^*)$

$$\|x^*\|_{JT_\infty(\mathscr{X}^*)} = \sup \left(\sum_{i\in I} \langle x^*, a_i \rangle^2 \right)^{1/2},$$

where the supremum is taken over all finite families $(a_i)_{i\in I}$ of atoms with disjoint supports and let $JT_\infty(\mathscr{X}^*)$ be the subset of $l^2(\mathscr{X}^*)$ consisting of the elements x^* for which $\|x^*\|_{JT_\infty(\mathscr{X}^*)} < +\infty$. Finally define $J_*T_\infty(\mathscr{X})$ as the closure of $l_2(\mathscr{X})$ in the dual of $JT_\infty(\mathscr{X}^*)$.

For each n let Y_n be the subspace of $J_*T_\infty(\mathscr{X})$ spanned by the elements on the nth level of the tree, i.e. $Y_n = [j_t(X_t)]_{|t|=n}$. The following properties were established in [GM2, §IV].

(1) Y_n is isometric to the l_2-sum of $\{j_t(X_t) : t \in T_\infty, |t| = n\}$.
(2) $(Y_n)_n$ is a monotone shrinking Schauder decomposition of $J_*T_\infty(\mathscr{X})$.
(3) $(Y_n)_n$ is a complemented boundedly complete skipped blocking decomposition of $J_*T_\infty(\mathscr{X})$.
(4) There exists a G_δ-embedding of $J_*T_\infty(\mathscr{X})$ into $(\sum_{n=1}^\infty \oplus Y_n)_{l_2}$.

For our present purpose, we shall add the following

Proposition IV.9. (i) *If the set $D_t = \{d_{t,i}\}_{i=1}^\infty$ consists of a normalized shrinking monotone basis of X_t for each t in T_∞, then the set $D = \{j_t(d_{t,i}) : t \in T_\infty, i \in \mathbf{N}\}$ forms a normalized shrinking monotone Schauder basis for $J_*T_\infty(\mathscr{X})$.*

(ii) *If the injections $S_t : X_t \to c_0$ defined by $S_t(\sum_{i=1}^\infty \lambda_i d_{t,i}) = (\lambda_i)_{i=1}^\infty$ are Tauberian for each t, then $(Y_n)_{n=1}^\infty$ is a JT_∞-type decomposition for $J_*T_\infty(\mathscr{X})$.*

(iii) *If in addition X_t has C.P.C.P for each $t \in T_\infty$, then $J_*T_\infty(\mathscr{X})$ has uniform C.P.C.P.*

Proof. (i) Enumerate D in a way that is compatible with the partial order: $j_t(d_{t,i}) \geq j_s(d_{s,j})$ if $(t > s)$ or $(t = s$ and $i \geq j)$. That is we enumerate D as $(d_n)_{n=1}^\infty = (j_{t_n}(d_{t_n,i_n}))_{n=1}^\infty$ such that for $m \geq n$ we cannot have $j_{t_m}(d_{t_m,i_m}) \leq j_{t_n}(d_{t_n,i_n})$. It is clear that this makes a monotone basic sequence out of the elements of D. Since the latter set spans a dense subspace of $J_*T_\infty(\mathscr{X})$, we conclude that the elements of D form a basis. That the basis is shrinking follows from (2).

To prove (ii) and (iii) we need the following lemmas which consist of appropriate modifications of arguments used in [GM2]. The first is contained in Lemma III.2 of [GM2]. We shall state it without proof.

Lemma IV.10. *Suppose* $x = (x_t)_{t\in T_\infty}$ *(with* $x_t \in X_t$ *for every* $t \in T_\infty$*) defines a bounded linear functional on* $JT_\infty(\mathscr{X}^*)$*. If* x *does not belong to* $J_*T_\infty(\mathscr{X})$*, then there exist a branch* γ *and a real* $\alpha > 0$ *such that for all* $k > 0$*,* $\|Q_k x\| = \|Q_k x'\| > \alpha$*. Here* x' *is the restriction of* x *to* γ *and* Q_k *is the projection on the subspace spanned by the vectors supported on* $\{s \in \gamma : |s| \geq k\}$*.*

The following is a refinement of Proposition III.1.c in [GM2].

Lemma IV.11. *Let* γ *be a branch of* T_∞*,* $(n_k)_{k=0}^\infty$ *a sequence of integers such that* $n_k + 1 < n_{k+1}$ *for every* $k \geq 0$ *and let* $(u_k)_{k=0}^\infty$ *be a sequence of norm-one vectors in* $J_*T(\mathscr{X})$ *with* $(u_k)_{k=0}^\infty$ *supported by* $S_k = \{\gamma(n_k+1), \dots, \gamma(n_{k+1}-1)\}$*.*
Let $\varepsilon > 0$ *and* $(w_k)_{k=0}^\infty$ *be elements in* $X_{\gamma(n_k)}$ *such that: for all* $k \in \mathbf{N}$*,*

$$|\langle w_k, d^*_{\gamma(n_k+1)}\rangle| < \varepsilon 2^{-k}. \tag{1}$$

Setting $x_k = j_{\gamma(n_k)}(w_k) + u_k$ *we then have* $\|\sum_{k=0}^K \alpha_k x_k\| \geq 1/4 - \varepsilon$ *for every family* $(\alpha_k)_{k=1}^k$ *of real numbers verifying* $\sum_{k=0}^K \alpha_k^2 = 1$*.*

Here $d^*_{\gamma(u_k+1)}$ denotes the element of $X^*_{\gamma(n_k)}$ biorthogonal to the element $d_{\gamma(n_k+1)}$ of the basis $(d_{\gamma(n_k),i})_{i=1}^\infty$ of $X_{\gamma(n_k)}$. Note that (1) holds whenever we have for all $k \in \mathbf{N}$:

$$\|S_{\gamma(n_k)} w_k\|_{c_0} < \varepsilon \cdot 2^{-k}. \tag{2}$$

Proof. We follow closely the arguments of [GM2, Proposition III.1]: Find (y_k^*) in $JT_\infty(\mathscr{X}^*)$, $\|y_k^*\| = 1$, $\langle y_k^*, u_k\rangle = 1$ such that y_k^* is supported by S_k. Let

$$\rho_k = \sum_{j=n_k+1}^{n_{k+1}-1} \langle y_k^*(\gamma(j)), d_{\gamma(j+1)}\rangle = \langle y_k^*, a_{T_k,0}\rangle,$$

where $T_k = S_k \cup \{\gamma(n_k)\} \cup \{\gamma(n_{k+1})\}$. We have $|\rho_k| \leq 1$. Let $v_k^* = \rho_k d^*_{\gamma(n_k+1)}$ and consider $z_k^* = -j_{\gamma(n_k)} v_k^* + y_k^* \in JT_\infty(\mathscr{X}^*)$. We have $\|z_k^*\| \leq 2$ and $\langle z_k^*, a_{T_k,0}\rangle = 0$. It follows that for every atom a:

$$\left\langle \sum_{k=1}^K \alpha_k z_k^*, a\right\rangle = \left\langle \sum_{k=0}^K \alpha_k z_k^*, a_1 + a_2\right\rangle,$$

where a_i, $i = 1, 2$, are atoms with support contained in some segment $\{\gamma(n_k), \dots, \gamma(n_{k+1} - 1)\}$, therefore

$$\left\| \sum_{k=0}^K \alpha_k z_k^* \right\| \leq 2 \left(\sum_{k=1}^K \alpha_k^2 \|z_k^*\|^2 \right)^{1/2} \leq 4.$$

Finally

$$\begin{aligned}4\cdot\left\|\sum_{k=0}^{K}\alpha_k x_k\right\| &\geq \left\langle \sum_{k=0}^{K}\alpha_k z_k^*, \sum_{k=0}^{K}\alpha_k(u_k + j_{\gamma(n_k)}(w_k))\right\rangle \\ &\geq 1-\left|\left\langle \sum_{k=0}^{K}\alpha_k v_k^*, \alpha_k w_k\right\rangle\right| \\ &\geq 1-\left|\sum_{k=0}^{K}\alpha_k^2\rho_k\varepsilon 2^{-k}\right| \geq 1-\varepsilon.\end{aligned}$$

Proof of IV.9(ii). For each $n\in\mathbf{N}$, let S_n be the operator from $Y_n = (\sum_{|t|=n}\bigoplus X_t)_{l_2}$ into c_0, defined by $S_n((x_t)_t) = (S_t(x_t))_{|t|=n}$. It is easy to check that S_n is also a Tauberian operator. We shall now verify that $(Y_n, S_n)_{n=1}^{\infty}$ is a JT_∞-type decomposition for $(J_* T_\infty(\mathscr{X}), S)$ where S is the operator

$$(S_n)_{n=1}^{\infty}\colon J_* T_\infty(X)\to c_0.$$

Note first that if $x = (x_t)_{t\in T_\infty}$ (with $x_t\in X_t$ for every $t\in T_\infty$) defines a bounded linear functional on $JT_\infty(\mathscr{X}^*)$ that verifies

$$\liminf_{n\to\infty}\|S_{\gamma(n)}(x_{\gamma(n)})\|_{c_0} = 0$$

for each infinite branch $\gamma = (\gamma(n))_{n=1}^{\infty}$ of T_∞, then x belongs to $J_* T_\infty(\mathscr{X})$. Indeed, if not, we can find by Lemma IV.10, a branch γ and a real scalar $\alpha > 0$ such that $\|Q_k x\| = \|Q_k x'\| > \alpha$, where x' is the restriction of x to γ. Since $\liminf_{n\to\infty}\|S_{\gamma(n)}x'_{\gamma(n)}\| = 0$ then, for any $\varepsilon > 0$, x' appears as a sum $\sum_{k=0}^{\infty} x'_k$ where $x'_k = j_{\gamma(n_k)}(w_k) + u_k$, u_k being supported by segments $S_k = \{\gamma(n_k+1), \dots, \gamma(n_{k+1}-1)\}$ $(n_k + 1 < n_{k+1})$ and w_k being elements in $X_{\gamma(n_k)}$ verifying for each k, $|\langle w_k, d^*_{\gamma(n_k+1)}\rangle| < \varepsilon 2^{-k}$. Since the partial sums $s_K = \sum_{k=0}^{K} x'_k$ are norm bounded, Lemma IV.11 implies that $(s_k)_k$ is actually norm convergent. This clearly contradicts the fact that $\|Q_k x'\| \geq \alpha > 0$ for all k.

Suppose now x^{**} is any element in the dual of $JT_\infty(\mathscr{X}^*)$ verifying $\pi_n^{**}\circ S^{**}x^{**}\in c_0$. Since $\pi_n^{**}\circ S^{**} = S_n^{**}\circ P_n^{**}$, where P_n is the projection on Y_n, and since S_n is a Tauberian embedding, we can find x_n in Y_n so that $S_n x_n = \pi_n^{**}\circ S^{**}x^{**} = S_n^{**}\circ P_n^{**}x^{**}$. It follows that x^{**} has all its coordinates x_t in X_t. If now $\liminf_{n\to\infty}\|\pi_n^{**}\circ S^{**}x^{**}\| = 0$, then $\liminf_{n\to\infty}\|S_{\gamma(n)}(x_{\gamma(n)})\|_{c_0} = 0$ for each infinite branch γ, hence $x^{**}\in J_* T_\infty(\mathscr{X})$ by the above observation. This finishes the proof of (ii).

(iii) follows immediately from (ii) and Theorem IV.5.

Proof of Theorem IV.8. Start with $J_* T_{\infty,0} = l_2, S\colon l_2\to c_0$ the canonical injection and $D = \{e_n\}_n$ the unit vector basis of l_2. Note that $(l_2, S, \{e_n\}_n)$ verify all the hypothesis in Proposition IV.9. Consider now the space $J_* T_{\infty,1} = J_* T_\infty(X_t, D_t; t\in T_\infty)$ where each $X_t = l_2$, $D_t = \{e_n\}_n$ and $S_t = S$. By the above proposition, we get that $(J_*T_{\infty,1}, D = \bigcup_{t\in T_\infty} j_t(D_t), S = (S_t)_{t\in T_\infty})$ also verify the same hypothesis. By induction, we can define $J_*T_{\infty,n+1} = J_*T_\infty(X_t, D_t; t \in T_\infty)$ where $X_t = J_*T_{\infty,n}$, and D_t is the set of its basis elements.

Assertion (b) follows now from IV.9(i). (d) is implied by (ii) and (e) by (iii). (f) is a consequence of (4) which was proved in [GM2, Proposition III.9]. (g) was proved for $n = 1$ in [GM1]. The rest follows from (d) since $J_*T_{\infty,1}$ embeds in all $J_*T_{\infty,n}$ for $n \geq 1$. For (h) note that if for $n \geq 2$, $J_*T_{\infty,n}$ has a complemented boundedly complete skipped blocking decomposition into reflexive subspaces $(E_k)_k$, then by [GM2, Proposition III.9], there exists a G_δ-embedding S from $J_*T_{\infty,n}$ into $(\sum_{k=1}^\infty \bigoplus E_k)_{l_2}$. But the latter is reflexive and separable, hence if R is any one-to-one operator from $(\sum_{n=1}^\infty \bigoplus E_n)_{l_n}$ into l_2 we get that $RS\colon J_*T_{\infty,n} \to l_2$ is a G_δ-embedding which contradicts (f).

V. About the C_* P.C.P and the R.N.P.—A counterexample

The following notion on dual Banach spaces was introduced and studied in [DGHZ]:

A dual Banach space X^* is said to have the *weak-star Convex Point of Continuity Property* (abbreviated C_* P.C.P) if every weak star compact convex subset C in X^* has a point of continuity for the Identity: $(C, w^*) \to (C, \| \|)$.

Note that this is the "dual version" of the C.P.C.P (the weak-star topology replacing the weak topology). On the other hand, if one considers analogously the "dual version" of the P.C.P (by leaving out the word *convex* in the above definition) then one recovers the R.N.P in view of the results of Stegall [St2].

In [GMS1] and §IV, we gave examples of Banach spaces with the C.P.C.P but failing the P.C.P. We show in the following that the "dual versions" of these notions are also different, thus answering negatively a question of G. Godefroy.

This section is devoted to the proof of the following.

Theorem V.1. *The dual JT^* of James-tree space has the C_* P.C.P while failing the R.N.P.*

Let JT denote the James tree space as defined in [LS] over the binary tree $T = \bigcup_{n=1}^\infty \{0,1\}^n$. Denote $\Delta = \{0,1\}^{\mathbf{N}}$ which we identify with the set of infinite branches of γ of T and let $\pi_w\colon JT^* \mapsto l^2(\Delta)$ be the quotient map which associates to each $x^* \in JT^*$ the limit values along the branches $\gamma \in \Delta$, i.e. $\pi_w(x^*) = (\lim_{n\mapsto\infty} x^*(\gamma(n)))_{\gamma\in\Delta}$. Denote by $\pi_n\colon JT^* \mapsto JT^*$ the restriction map to the nth level $T_n = \{0,1\}^n$ of T and, for $n \leq m$, $\pi[n,m] = \pi_n + \cdots + \pi_m$ the restriction map to $T_n \cup \cdots \cup T_m$.

The next lemma follows easily from the analysis in [LS] (see also [GMS, Lemma 1.2]).

Lemma V.2. *For each x^* in JT^* we have:*

$$\|\pi_w(x^*)\|_{l^2(\Delta)} = \lim_{n\mapsto\infty} \|\pi_n(x^*)\|_{JT^*} = \lim_{n\mapsto\infty} \sup_{m\geq n} \|\pi_{[n,m]}(x^*)\|_{JT^*}.$$

In particular there exist no x^* in JT^*, scalars $0 \le \beta < \alpha$ and an increasing sequence of integers $(n_k)_{k=1}^{\infty}$ such that for all k:

(i) $\|\pi_{n_k}(x^*)\| \le \beta$,

(ii) $\|\pi_{[n_k, n_{k+1}]}(x^*)\| \ge \alpha$.

Lemma V.3. *Let C be a bounded (not necessarily convex) subset of JT^* and $\varepsilon > 0$ such that* $\operatorname{diam}(\pi_w(C)) < \varepsilon$. *Then there is a weak-star relatively open subset V of C such that* $\operatorname{diam}(V) < 10\varepsilon$.

Proof. Modulo a translation of C we can assume without loss of generality that

$$\pi_w(C) \subseteq \varepsilon\,\mathrm{ball}(l^2(\Delta)). \tag{1}$$

Assuming that the assertion of the lemma is false, we construct inductively a decreasing sequence $(V_k)_{k=1}^{\infty}$ of relatively weak-star open subsets of C and increasing sequences $(n_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $n_1 < m_1 < n_2 < m_2 < \cdots$ such that, for $k \in \mathbf{N}$ and $y^* \in V_k$

(i) $\|\pi_{n_i}(y)\| < 2\varepsilon$ for $i \le k$,

(ii) $\|\pi_{[n_i, m_i]}(y)\| > 4\varepsilon$ for $i < k$.

The construction is straightforward: Let $V_0 = C$ and suppose V_i, n_i, m_i are defined for $i < k$.

Choose $x_k^* \in V_{k-1}$. In view of (1) and Lemma V.2 there is an $n_k \in \mathbf{N}$, $n_k > m_{k-1}$, such that $\|\pi_{n_k}(x_k^*)\| < 2\varepsilon$.

Let U_k be a relative weak-star neighborhood of x_k^* in C, $U_k \subseteq V_{k-1}$, such that, for every $y^* \in U_k$, $\|\pi_{n_k}(y^*)\| < 2\varepsilon$ and for $y^*, z^* \in U_k$, $\|\pi_{[0, n_k]}(y^* - z^*)\| < \varepsilon$.

As the diameter of U_k is greater than 10ε there is $y_k^* \in U_k$ and $m_k > n_k$ such that $\|\pi_{[n_k, m_k]}(y_k^*)\| > 4\varepsilon$. Let V_k be a weak-star relatively open subset of U_k such that, for every $y^* \in V_k$, $\|\pi_{[n_k, m_k]}(y^*)\| > 4\varepsilon$.

This completes the induction step. By compactness the intersection $\bigcap_{k=1}^{\infty} \overline{V_k}^*$ is nonempty, but any x^* in such an intersection will contradict Lemma V.1. □

Remark V.4. Given a bounded set $C \subseteq JT^*$, one easily verifies that, for $\varepsilon > 0$ there is a slice S of C, determined by some $x^{**} \in JT^{**}$, such that $\operatorname{diam}(\pi_w(S)) < \varepsilon$. Hence by Lemma V.3 there is a weak-star open set V of JT^* such that the diameter of $V \cap S$ has diameter less than 10ε. Noting that $V \cap S$ is a relatively weakly open subset of C we see that Lemma V.3 implies in particular the result of G. Edgar and R. Wheeler [EW] stating that JT^* has the P.C.P.

The proof of the next easy lemma is left to the reader.

Lemma V.5. *Let D be a convex subset of the unit-ball of Hilbert space and let $\alpha > 0$. If* $\operatorname{diam}(D) \ge \alpha$, *then* $\sup\{\|x\| : x \in D\} \ge \inf\{\|x\| : x \in D\} + \alpha^2/8$.

Now we can prove that: *JT^* has C_* P.C.P.*

Indeed, let C be a convex subset of the unit ball of JT^*. If the claim were false, then by Lemma V.3 we could find $\alpha > 0$ such that for every weak-star relatively open subset V of C, $\operatorname{diam}(\pi_w(V)) > \alpha$. By Lemma V.5 there is $\beta > 0$ such that for every relatively weak-star open $V \subseteq C$

$$(2) \qquad \sup\{\|\pi_w(x^*)\| : x^* \in V\} \geq \inf\{\|\pi_w(x^*)\| : x^* \in V\} + \beta .$$

We now construct inductively a decreasing sequence $(V_k)_{k=1}^{\infty}$ of relatively weak-star open subsets of C, an increasing sequence $(r_k)_{k=1}^{\infty}$ in $[0,1]$ and a sequence $(n_k)_{k=1}^{\infty}$ in $\mathbf{N}$, such that

(i) $r_{k+1} - r_k > \beta/3$ for $k \in \mathbf{N}$,

(ii) $|\,\|\pi_{n_i}(y^*)\| - r_i| < \beta/10$ for $i \leq k$ and $y^* \in V_k$.

Let $V_0 = C$, $n_0 = 0$, $r_0 = 0$ and suppose V_i, n_i and r_i are chosen for $i < k$. We infer from (2) that there is $x_k^* \in V_{k-1}$ and $r_k \in [0,1]$ such that $\|\pi_{n_w}(x_k^*)\| = r_k$ and $r_k - r_{k-1} > \beta/3$. By Lemma V.2 there is $n_k > n_{k-1}$ such that $|\,\|\pi_{n_k}(x_k^*)\| - r_k| < \beta/10$ and we may choose a relative weak-star neighborhood V_k of x_k^* in C with $V_k \subseteq V_{k-1}$ such that for every $y^* \in V_k$, $|\,\|\pi_{n_k}(y^*)\| - r_k| < \beta/10$. This completes the induction step. Again $\bigcap_{k=1}^{\infty} \overline{V}_k^*$ is nonempty, whence we arrive—in view of Lemma V.2—to a contradiction.

To finish the proof of the theorem, it is enough to recall that JT^* is not separable [LS], hence it does not have the R.N.P. [St2].

Remark V.6. The following duality theorem was proved in [DGHZ]: The dual X^* of a separable Banach space has C_* P.C.P if and only if X is a *Phelps space*: that is every equivalent Gateaux differentiable norm on X is Fréchet differentiable on a dense set. On the other hand, it is well known [St2] that X^* has R.N.P if and only if X is an *Asplund space*: that is *all* equivalent norms on X are Fréchet differentiable on a dense set. The above result shows that JT is then a Phelps space which fails to be an Asplund space.

VI. A strongly regular Banach space failing the C.P.C.P

This section is devoted to the construction of the following counterexample. A set which is strongly regular while failing C.P.C.P was constructed simultaneously in [AOR]. The present example was also presented in [S3].

Theorem VI.1. *There exists a Banach space—denoted S_*T_∞—with the following properties:*

(a) *S_*T_∞ fails the C.P.C.P.*

(b) *S_*T_∞ has no JT_∞-type decomposition into reflexive subspaces.*

On the other hand

(c) *Its dual-denoted ST_∞-is separable.*

(d) *The quotient ST_∞^*/S_*T_∞ is reflexive.*

(e) *Its double dual ST_∞^* is strongly regular.*

(f) *S_*T_∞ has a complemented boundedly complete skipped blocking decomposition into Hilbert spaces.*

(g) *There exists a G_δ-embedding of S_*T_∞ into l^2.*

We start by constructing a set which is strongly regular but failing the C.P.C.P. The space $S_* T_\infty$ will be obtained by a suitable interpolation procedure.

Let again T_∞ be the infinite tree $\bigcup_{n=0}^{\infty} \mathbf{N}^n$ and denote by ϕ the *origin* of the tree. Let T_∞^n be the nth level of the tree, i.e. $T_\infty^n = \{t \in T_\infty : |t| = n\}$ where $n \in \mathbf{N}$. Let $X = (\sum_{n=0}^{\infty} \bigoplus l^2(T_\infty^n))_{c_0}$, that is the Banach space

$$X = \left\{ x = (x_t)_{t \in T_\infty} : \|x\|^2 = \sup_n \sum_{|t|=n} |x_t|^2 < \infty \text{ and } \lim_n \sum_{|t|=n} |x_t|^2 = 0 \right\}.$$

Let K be the subset of $X^{**} = (\sum_{n=0}^{\infty} \bigoplus l^2(T_\infty^n))_{l_\infty}$ consisting of all functions z on T_∞ satisfying:

(i) $z_\phi = 1$,

(ii) $z_t \geq 0$ for all t in T_∞,

(iii) $z^2_{(n_1, n_2, \ldots, n_k)} \geq \sum_{n_{k+1}=1}^{\infty} z^2_{(n_1, n_2, \ldots, n_{k+1})}$ for all $(n_1, n_2, \ldots, n_k)$ in T_∞.

Proposition VI.2. *The set K is a w^*-compact convex subset of X^{**} verifying the following properties*:

(a) *$K_0 = K \cap X$ is a closed convex bounded subset of X failing the C.P.C.P.*

(b) *The expression $(x|y) = \lim_{n \to \infty} \sum_{|t|=n} x_t y_t$ is well defined for each x, y in K and extends to all vectors in the norm closed linear span Y of K in X^{**}.*

(c) *If π denote the quotient map from X^{**} onto $X^{**}|X$, we have $\|\pi(x)\|_{X^{**}/X} = (x|x)^{1/2}$ for all x in Y. Hence $Y|X$ is isometric to Hilbert space.*

(d) *K is strongly regular.*

Proof. (a) We shall show that every relatively weakly open subset of K_0 has diameter equal to $\sqrt{2}$. For that let $x \in K_0$ and let A be a finite subset of T_∞. For $\varepsilon > 0$, consider the weak neighborhood of x in K_0

$$V_{A,\varepsilon}(x) = \{\xi \in K_0 : |x_t - \xi_t| < \varepsilon \ \forall t \in A\}.$$

As the elements with finite support in $X^* = (\sum_{n=0}^{\infty} \bigoplus l^2(T_\infty^n))_{l_1}$ are dense in X^*, the family $V_{A,\varepsilon}(x)$ forms a relative weak neighborhood base of x in K_0. Hence it is enough to show that for each x, A, ε as above

$$\operatorname{diam}(V_{A,\varepsilon}(x)) = \sqrt{2}.$$

There is no loss in generality in assuming that A is *full*, i.e., $t \in A$ and $s \leq t$ implies $s \in A$.

Find n large enough such that for each $t \in A$, the length $|t|$ is less than n.

We shall first show that there is $\xi \in K_0$ such that $\xi_t = x_t$ for $t \in A$ (whence in particular $\xi \in V_{A,\varepsilon}(x)$) and $\sum_{|t|=n} |\xi|^2 = 1$.

Fix $\bar{k} \in \mathbf{N}$ such that it does not occur in any component of $t = (k_1, \ldots, k_l)$, where $t \in A$. We shall define ξ inductively on the levels T_∞^i. Let $\xi_\phi = 1$. For $i = 1$ let

$$m_\phi^2 = \sum_{t \in A, |t|=1} x_t^2.$$

By assumption $m_\phi^2 \leq 1$. Define ξ on T_∞^1 by

$$\begin{cases} \xi_t = x_t & \text{if } t \in A, \ |t| = 1, \\ \xi_{\{\bar{k}\}} = (1 - m_\phi^2)^{1/2}, & \\ \xi_t = 0 & \text{elsewhere on } T_\infty^1. \end{cases}$$

Let $B_1 = (T_1 \cap A) \cup \{\bar{k}\}$, which is a finite subset of T_∞^1.

Suppose now that ξ is defined on $T_\infty^1, \ldots, T_\infty^{i-1}$ and that there is a subset $B_{i-1} \subseteq T_\infty^{i-1}$ such that ξ vanishes outside of $T_\infty^{i-1} \backslash B_{i-1}$. For every $s \in B_{i-1}$ let

$$m_s^2 = \sum_{t \in A, |t|=i, t \geq s} x_t^2.$$

By assumption, $m_s^2 \leq x_s^2$. Define ξ on T_∞^i by

$$\begin{cases} \xi_t = x_t & \text{if } t \in A, \ |t| = i, \\ \xi_{\{s, \bar{k}\}} = (1 - m_s^2)^{1/2} & \text{if } s \in B_{i-1}, \\ \xi_t = 0 & \text{elsewhere on } T_\infty^i. \end{cases}$$

Define $B_i = (T_\infty^i \cap A) \cup \{\{s, \bar{k},\} : s \in B_{i-1}\}$.

Continue this inductive process up to $i = n$ and define ξ_t to be zero on T_∞^i for $i > n$. One easily verifies that ξ is an element of K_0 with the desired properties.

Now let $\xi^{(1)}$ and $\xi^{(2)}$ be such that:

$$\begin{cases} \xi_t^{(1)} = \xi_t^{(2)} = \xi_t & \text{for } t \in T \backslash T_\infty^{n+1}, \\ \xi_{\{s,1\}}^{(1)} = \xi_{\{s,2\}}^{(2)} = \xi_s & \text{for } s \in B_n, \\ \xi_t^{(1)} = 0 & \text{for } t \in T_\infty^{n+1}, \ t \text{ not of the form } \{s, 1\} \text{ with } s \in B_n, \\ \xi_t^{(2)} = 0 & \text{for } t \in T_\infty^{n+1}, \ t \text{ not of the form } \{s, 2\} \text{ with } s \in B_n. \end{cases}$$

Again $\xi^{(1)}$ and $\xi^{(2)}$ are in K_0, they coincide with x on A and

$$\|\xi^{(1)} - \xi^{(2)}\| \geq \left(\sum_{t \in T_\infty^{n+1}} |\xi_t^{(1)} - \xi_t^{(2)}|^2 \right)^{1/2} = \sqrt{2},$$

as $\xi^{(1)}$ and $\xi^{(2)}$ have disjointly supported l^2-masses 1 on the level T_∞^{n+1}.

The reverse estimate is implied by observing that the whole set K_0 has diameter $\sqrt{2}$. Hence $\operatorname{diam}(V_{A,\varepsilon}(x)) = \sqrt{2}$.

The proof of (b) consists of a simple application of the Cauchy-Schwarz inequality:

$$\begin{aligned}\sum_{|t|=n+1} x_t y_t &= \sum_{(k_1,\dots,k_n)\in T^n} \sum_{k_{n+1}=1}^{\infty} x_{(k_1,\dots,k_{n+1})} y_{(k_1,\dots,k_{n+1})} \\ &\le \sum_{(k_1,\dots,k_n)\in T^n} \left(\sum_{k_{n+1}=1}^{\infty} x^2_{(k_1,\dots,k_{n+1})} \sum_{k_{n+1}=1}^{\infty} y^2_{(k_1,\dots,k_{n+1})} \right)^{1/2} \\ &\le \sum_{(k_1,\dots,k_n)\in T^n} x_{(k_1,\dots,k_{n+1})} y_{(k_1,\dots,k_{n+1})} = \sum_{|t|=n} x_t y_t .\end{aligned}$$

Hence $(x|y)$ exists as it is the limit of a decreasing sequence of positive numbers.

In particular, for $x \in K$ the norms of x restricted to T^n_∞ converge towards $(x|x)^{1/2}$.

Now let Y be the norm-closed linear span of K in X^{**} equipped with the norm inherited from X^{**}. Then for $\xi, \eta \in Y$ we still may define

$$(\xi|\eta) = \lim_{n\to\infty} \sum_{|t|=n} \xi_t \eta_t .$$

Indeed if $\xi = \sum_{i=1}^{k} \lambda_i x_i$ and $\eta = \sum_{j=1}^{l} \mu_j y_j$ are in the linear span of K, then

$$(\xi|\eta) = \sum_{i=1}^{k} \sum_{j=1}^{l} \lambda_i \mu_j \lim_{n\to\infty} \sum_{|t|=n} x_{i,t} y_{j,t} .$$

To go from the linear span of K to the norm-closure in X^{**} simply observe that for $\xi, \eta \in X^{**}$ and $n \in \mathbf{N}$

$$\sum_{|t|=n} \xi_t \eta_t \le \|\xi\|_{X^{**}} \|\eta\|_{X^{**}}$$

which easily implies assertion (b).

To prove (c) of VI.2, note first that Y contains X as the span of K_0 is norm-dense in X. The above formula implies that the norm in Y/X is defined by an inner product, hence Y/X is isometric to Hilbert-space.

For (d) we first note that l^1 does not embed in the space Y since it clearly does not embed in neither $X = (\sum_{n=1}^{\infty} \oplus l^2(T_n))_{c_0}$ nor $Y/X = l^2(\Gamma)$. That K is strongly regular follows from the following general fact.

Proposition VI.3. *Let K be a w^*-compact convex subset of the dual of a separable Banach space Z. The following properties are then equivalent:*

(a) *K is strongly regular.*
(b) *There exists no family $(e_s)_{s\in[0,1]}$ in K that is equivalent to the unit vector basis of $l^1([0,1])$.*

This proposition is well known in the case where K is the unit ball of Z^* (see [P and B2]). This "local version" can be obtained with the same methods without any additional difficulty. We refer the reader to [GGMS and S3] for more details. □

We now proceed with the construction of the space. Denote by W the symmetric closed convex hull of K_0 in X. The space S_*T_∞ will be the interpolation space associated to W in X by the method of [DFJP]. That is

$$S_*T_\infty = \left\{ x \in X:\ \|| x \|| = \left(\sum_{n=1}^{\infty} \|x\|_n^2 \right)^{1/2} < \infty \right\},$$

where $\|x\|_n$ is the gauge of the set $U_n = 2^n W + 2^{-n}\,\mathrm{ball}(X)$ for each $n = 1, 2, \dots$. Let $j: S_*T_\infty \to X$ be the identity mapping.

We shall now verify the properties (a)–(g) of S_*T_∞ claimed in Theorem VI.1.

To prove (a) note that $K_0 \subseteq j(\mathrm{ball}(S_*T_\infty)) \subseteq \bigcap_n (2^n W + 2^{-n}\,\mathrm{ball}(X))$. It follows that $L_0 = j^{-1}(K_0)$ is a closed convex bounded subset of S_*T_∞. Since $j^*(X^*)$ is norm dense in $(S_*T_\infty)^* = ST_\infty$ [DFJP, Lemma 1(iii)], we get that j defines a homeomorphism between L_0 and K_0 for the respective weak topologies. So for a relatively weak open subset U of L_0, $j(U)$ is relatively weakly open in K_0, hence $\mathrm{diam}_X(j(U)) = \sqrt{2}$ and therefore $\mathrm{diam}_{S_*T_\infty}(U) \geq \sqrt{2}\|j\|^{-1}$, which clearly shows that S_*T_∞ fails the C.P.C.P.

(b) follows immediately from (a) and Theorem IV.5, which in fact shows that S_*T_∞ does not have a JT_∞-type decomposition into spaces with C.P.C.P.

(c) is an immediate consequence of the fact that $X^* = (\sum_{n=0}^{\infty} \bigoplus l^2(T_n))_{l_1}$ is separable and that $j^*(X^*)$ is norm dense in ST_∞.

(d) By Lemma 1(xiii) of [DFJP], the space ST_∞^*/S_*T_∞ is isometric to a subspace of the space Z_0 obtained by applying the interpolation construction to $X_0 = X^{**}/X$ and $W_0 = \pi(\overline{W}^*)$, where $\overline{W}^*$ is the w^*-closure of W in X^{**}. Since $\overline{W}^* \subseteq \mathrm{conv}(K \cup (-K))$ we have that $\pi(\overline{W}^*)$ is contained in the Hilbert space Y/X. It follows that W_0 is relatively weakly compact and Z_0 is reflexive [DFJP, Lemma 1.iv]. In fact, it can be shown that in this case ST_∞^*/S_*T_∞ is isometric to Hilbert space.

(e) Note that l_1 embeds neither in S_*T_∞ (since its dual is separable) nor in ST_∞^*/S_*T_∞ (since it is reflexive). It follows that l_1 does not embed in ST_∞^*. In particular, the latter space is strongly regular by Proposition VI.3.

(f) For each n, let S_n be the natural projection of X onto $(\sum_{m<n} \bigoplus l^2(T_m))$. Note that $(S_n)_n$ is a monotone shrinking Schauder decomposition for X such that $S_n(W) \subseteq W$ for each n. It follows from [DFJP, Lemma 1.ix] that the projections $P_n = j^{-1} S_n j$ form a monotone shrinking Schauder decomposition for S_*T_∞. Note that j is an isometry between $E_n = (P_n - P_{n-1})(S_*T_\infty) =$

$j^{-1}(l^2(T^n_\infty))$ and $l^2(T^n_\infty)$. It remains to show that $(E_n)_n$ is a boundedly complete skipped blocking decomposition for S_*T_∞. We shall start by showing that

$$\begin{aligned}\operatorname{ball}(S_*T_\infty) &= \left\{ z^{**} \in \operatorname{ball}(ST^*_\infty) \colon \lim_{n\to\infty} \sum_{|t|=n} j^{**}(z^{**})_t^2 = 0 \right\} \\ &= \left\{ z^{**} \in \operatorname{ball}(ST^*_\infty) \colon \liminf_{n\to\infty} \sum_{|t|=n} (j^{**}(z^{**})_t)^2 = 0 \right\}.\end{aligned}$$

First note that

$$\begin{aligned} j^{**}(\operatorname{ball}(ST^*_\infty)) &\subseteq \bigcap_n \overline{(2^n W + 2^{-n} B_X)}^{*} \\ &\subseteq \bigcap_n (2^n \operatorname{conv}(K \cup -K) + 2^{-n} B_{X^{**}}) \subseteq Y, \end{aligned}$$

so for each $z^{**} \in \operatorname{ball}(ST^*_\infty)$,

$$\lim_{n\to\infty} \sum_{|t|=n} j^{**}(z^{**})_t^2 = (j^{**}(z^{**})|j^{**}(z^{**}))$$

exists and hence the last two proposed expressions for $\operatorname{ball}(S_*T_\infty)$ are equal. On the other hand, if $z^{**} \in \operatorname{ball}(ST^*_\infty)$ and $\lim_{n\to\infty} \sum_{|t|=n} (j^{**}(z^{**})_t)^2 = 0$, then $j^{**}(z^{**}) \in X$, hence $z^{**} \in S_*T_\infty$ [DFJP, Lemma 1(iii)] (j is Tauberian!) and our claim is proved.

Assume now $(n_k)_k$ is a strictly increasing sequence of integers and for every $k \geq 0$ let z_k be an element of S_*T_∞ supported by $Z_k = E_{n_k+1} \oplus \cdots \oplus E_{n_{k+1}-1}$ such that the partial sums $(\sum_{k=0}^K z_k)_K$ are in the unit ball of S_*T_∞. Let z^{**} be a weak*-limit of $(\sum_{k=0}^K z_k)_K$ in the unit ball of ST^*_∞. Since $j^{**}(z^{**})$ vanishes on the level n_k for each k, we get that $\liminf_{n\to\infty} \sum_{|t|=n} j^{**}(z_t^{**})^2 = 0$. If follows from the above that z^{**} belongs to S_*T_∞ and that the series $(\sum_{k=0}^K z_k)_K$ converges in S_*T_∞. Hence $(Z_k)_k$ is a boundedly complete Schauder decomposition of its closed linear span.

(g) This follows immediately from (f) and Proposition III.4 of [GM2]. In this case the G_δ-embedding may be chosen to be $T \circ j$, where $T\colon (\sum_{n=0}^\infty \bigoplus l^2(T^n_\infty))_{c_0} \to (\sum_{n=0}^\infty \bigoplus l^2(T^n_\infty))_{l_2}$ is defined by $T((x_t)_{t\in T_\infty}) = (2^{-|t|} x_t)_{t \in T_\infty}$.

Remark VI.4. It is clear that the space S_*T_∞ contains no subspace isomorphic to l_1. On the other hand, we do not know whether it contains l_n^1's uniformly. In view of the recent work of Pisier and Xu [PX] one may ask whether S_*T_∞ has type 2.

Remark VI.5. Unlike S_*T_∞, the space J_*T_∞ was not constructed as an interpolation space [GM1]. However, one may use such a procedure to construct spaces with the same topological properties as J_*T_∞ and probably (as suggested by the work of Pisier-Xu mentioned above) with an improved "local structure". Indeed, first note that the natural injection $S\colon J_*T_\infty \to c_0(T_\infty)$ is clearly a Tauberian embedding (Remark I.9 and Proposition IV.9). Consider now the set L_0 of J_*T_∞ equal to the norm closed convex hull of the atoms $a_{S,0}$ where S is any segment starting at ϕ. Let L be its weak*-closure in JT_∞^*. It is clear that its image in $l_\infty(T_\infty)$ is the set of all functions z on T_∞ verifying:

(i) $z_\phi = 1$,
(ii) $z_t \geq 0 \ \forall t \in T$,
(iii) $z_{(n_1, n_2, \ldots, n_k)} \geq \sum_{n_{k+1}=1}^{\infty} z_{(n_1, n_2, \ldots, n_k, n_{k+1})} \ \forall (n_1, n_2, \ldots, n_k) \in T_\infty$.

Analogous to the case of S_*T_∞, we shall call this set K^2. It is easy to see that $K_0^2 = K^2 \cap c_0 = S(L_0)$. Since S is Tauberian, it follows immediately from what was established for J_*T_∞ in [GM1] and [GMS] that K_0^2 has C.P.C.P while failing the P.C.P. These properties of K_0^2 were also reproved in [AOR] where this set is studied under the name of STS.

As in the case of ST_∞, one can consider the interpolation space associated to K_0^2. This will coincide with the interpolation space $[J_*T_\infty, c_0]_{\theta,q}$, where $\theta = 1/2$ and $q = 2$. Since the injection from $[J_*T_\infty, c_0]_{\theta,q} \to c_0$ is also Tauberian, it is easy to check that this new space has the C.P.C.P and a boundedly complete skipped blocking decomposition into Hilbert spaces, while failing the P.C.P. Again, it is probable that $[J_*T_\infty, c_0]_{\theta,q}$ has better "local properties" than J_*T_∞ and we may ask whether it has type 2.

The same question applies to the space $[J_*T, c_0]_{\theta,q}$ (i.e. when T is finitely branching). Note that the same reasoning as above shows that it has the P.C.P but fails the R.N.P.

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